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A Risk–Neutral Characterization of Optimism and Pessimism, and its Applications*

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Abstract

This note gives a simple, but useful characterization of optimism and pessimism represented by a convex and concave shift of probability weighting functions, and applies it to two comparative static analysis.

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1 Introduction

Optimism and pessimism play a crucial role in decision–making under risk. Expected utility theory is the dominant theory to analyze decision–making under risk. However, it can not express optimism and pessimism. The widely used alternative to expected utility theory is rank dependent utility theory introduced by Quiggin (1981, 1982), and it can express optimism and pessimism. This note gives a simple, but useful characterization of optimism and pessimism represented by a convex and concave shift of probability weighting functions, and presents its comparative static applications.

In this note, optimism and pessimism are characterized by monotone likelihood ratio dominance between their corresponding risk–neutral decision weights. This characterization is expected to be useful for comparative static analysis in the following reasons. It is known that this characterization is useful to obtain sharp comparative static results in expected utility theory, *e.g.* Osaki (2005), Ohnishi and Osaki (2006a) and others.[1] As in Quiggin (1991), rank dependent expected utility can be regarded as "expected utility with respect to a transformed probability distribution". This observation suggests that we can also obtain them in rank dependent expected utility theory. Also, monotone likelihood ratio dominance is defined as log–supermodularity between compared probability density functions. It plays an important role in comparative static analysis under risk (Athey, 2002; Jewitt, 1987).

This note is related to previous studies about comparative static analysis in generalized expected utility theory including rank dependent expected utility theory.

[1] Ohnishi and Osaki (2006b) extended this characterization to nonexpected utility theory applying the property of log–supermodularity.

Quiggin (1991, 1995) determined conditions to preserve comparative static results in expected utility theory to those in rank dependent expected utility theory. Schlee (1994) claimed that monotone likelihood ratio dominance may not have the comparative static prediction under the generalized expected utility theory. The concern of these papers is the preservation of comparative static results from expected utility theory to rank dependent expected utility theory. In contrast, our concern is how shifts in probability weighting function representing optimism and pessimism influence economic behavior under risk.

The organization of our note is as follows. Section 2 briefly gives some preliminaries and provides a risk–neutral characterization of optimism and pessimism. In Section 3, we obtain two comparative static predictions applying the characterization in Section 2. Section 4 is a conclusion.

2 A risk–neutral characterization of optimism and pessimism

First of all, we give a representation of Rank Dependent Expected Utility (RDEU), a preference relation defined over discrete random variables. A discrete random variable is an outcome vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_S)$ with a corresponding probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_S)$. The probability is strictly positive, $p_s > 0$ and sum to one, $\sum_{s=1}^{S} p_s = 1$. Without loss of any generality, we assume that the outcomes are ranked in ascending order, $x_1 < x_2 < \ldots < x_S$. A Decision Maker (DM) has RDEU representation if the discrete random variable $(p_1, x_1; p_2, x_2; \ldots; p_S, x_S)$ is evaluated by

$$V(p_1, x_1; p_2, x_2; \ldots; p_S, x_S) := \sum_{s=1}^{S} d_s^q u(x_s), \tag{1}$$

where $\boldsymbol{d}^q = (d_1^q, d_2^q, \ldots, d_S^q)$ is a decision weight vector with respect to a Probability Weighting Function (PWF) q, and u is a utility function. The decision weight d_s^q is

$$d_s^q = q(p_1 + p_2 + \ldots + p_s) - q(p_1 + p_2 + \ldots + p_{s-1}), \tag{2}$$

and $d_1^q = q(p_1)$. For notational ease, the cumulative probability is denoted as $P_t = \sum_{s=1}^{t} p_s$. The probability weighting function q is increasing in P with $q(0) = 0$ and $q(1) = 1$. Expected utility representation corresponds to the linear PWF, $q(P) = P$. The utility function u is increasing in x. We note that concavity of the utility function is not necessary for our analysis. Outcomes are ranked in ascending order in our analysis, on the other hand, other many researches represent that they are ranked in descending order, $x_1 > x_2 > \ldots > x_S$. We can present an identical analysis in both outcome representations because of the duality of corresponding PWFs (Diecidue and Wakker; 2001).

To characterize optimism and pessimism, we define two notions. First, we define the risk–neutral decision weight corresponding to the PWF q as follows:

$$\hat{d}_s^q := \frac{d_s^q u'(x_s)}{\sum_{s=1}^{S} d_s^q u'(x_s)}. \tag{3}$$

It is clear that the risk–neutral decision weight is also the probability of outcome x_s, since it is strictly positive, $\hat{d}_s^q > 0$, and sums to one, $\sum_{s=1}^{S} \hat{d}_s^q = 1$. Next, we give a definition of “more optimistic” (“more pessimistic”) as follows:

Definition 2.1. A probability weighting function q is more optimistic (pessimistic) than r if there exists an increasing and convex (concave) function ϕ such that $q = \phi \circ r$.

We give a couple of explanations about optimism and pessimism. First, more optimistic (pessimistic) PWFs underweight (overweight) the cumulative probability of worse outcomes, $q(P) \leq (\geq)\ r(P)$ for all P. Second, pessimistic PWFs characterize the strong risk aversion, aversion to mean–preserving increase in risk (Rothschild and Stiglitz; 1970). This characterization was first demonstrated by Chew, Karni and Safra (1987). In a recent paper, Ryan (2006) obtained it under weaker conditions and also displayed a nice summary of risk aversion in RDEU.

Next, we define a stochastic dominance characterizing optimism and pessimism:

Definition 2.2. Let us consider two probability vectors $\boldsymbol{p}^1 = (p_1^1, p_2^1, \ldots, p_S^1)$ and $\boldsymbol{p}^2 = (p_1^2, p_2^2, \ldots, p_S^2)$. The probability vector $\boldsymbol{p}^2$ dominates $\boldsymbol{p}^1$ in the sense of monotone likelihood ratio dominance, if

$$\frac{p_t^2}{p_s^2} \geq \frac{p_t^1}{p_s^1} \tag{4}$$

for all $s < t$.

Monotone Likelihood Ratio Dominance (MLRD) is a stronger stochastic dominance than First–order Stochastic Dominance (FSD). The proof is in standard references of stochastic dominance *e.g.* Shaked and Shanthikumar (1994), Müller and Stoyan (2002) and others.

Theorem 2.1. A probability weighting function q is more optimistic (pessimistic) than r, if and only if the corresponding risk–neutral decision weight vector $\hat{\boldsymbol{d}}^q$ with respect to the probability weighting function q dominates (is dominated by) $\hat{\boldsymbol{d}}^r$ in the sense of monotone likelihood ratio dominance.

Proof. We only give the proof of the optimistic case, since the pessimistic case is similar. First, we obtain the following inequality by the definition of MLRD.

$$\frac{\hat{d}_t^q}{\hat{d}_s^q} = \frac{d_t^q u'(x_t)}{d_s^q u'(x_s)} \geq \frac{d_t^r u'(x_t)}{d_s^r u'(x_s)} = \frac{\hat{d}_t^r}{\hat{d}_s^r}, \tag{5}$$

for all $s \leq t$. It can be rewritten $d_t^q/d_s^q \geq d_t^r/d_s^r$.

We define $R_s := r(P_s)$. By noting that $R_s \leq R_t$ for all $s \leq t$ because of r increasing in P, the convexity of ϕ is equivalent to

$$\frac{\phi(R_t) - \phi(R_{t-1})}{R_t - R_{t-1}} \geq \frac{\phi(R_s) - \phi(R_{s-1})}{R_s - R_{s-1}}. \tag{6}$$

Since $q = \phi \circ r$, we have

$$\frac{q(P_t) - q(P_{t-1})}{q(P_s) - q(P_{s-1})} \geq \frac{r(P_t) - r(P_{t-1})}{r(P_s) - r(P_{s-1})}. \tag{7}$$

This can be rewritten $d_t^q/d_s^q \geq d_t^r/d_s^r$ by the definition of the decision weight.

Combining the above two discussions, we complete the proof. □

Since the identical PWF corresponds to the expected utility, we have the following corollary: a risk–neutral decision weight with respect to a convex (concave) probability weighting function dominates (is dominated by) that for the expected utility decision–maker. Quiggin (1995) pointed out that a convex (concave) PWF dominates (is dominated by) the original distribution function in the sense of MLRD. This is essentially the second part of the proof. This claim suggests that comparative static analysis of MLRD changes in the expected utility theory can also be applied to those of the shift to a more optimistic (pessimistic) PWF in RDEU.

3 Applications

In this section, we give two comparative static results applying Theorem 2.1. First, we demonstrate that more optimistic (pessimistic) representative investors lead to icreases (decreases) in equilibrium asset prices. Second, we display that more optimistic (pessimistic) decision makers behave in a more risk–tolerant (risk–averse) manner under background risk when utility functions exhibit decreasing absolute risk aversion.

3.1 Asset price

We consider a static version of Lucas (1978), a two–date pure exchange economy with a representative investor. The representative investor has the RDEU with a PWF q and an increasing utility function u. There are two assets traded in the asset market, risk–free and risky assets. The risk–free asset is the numeraire, and its (gross) return can be normalized to one, without loss of any generality. The return of the risky asset is a discrete random variable $(p_1, x_1; p_2, x_2; \ldots; p_S, x_S)$, and its price is π^q. The endowment consists of w units of the risk–free asset and one unit of the risky asset. The investor determines the investment in the two assets to maximize the RDEU from the final wealth. The investments of the risk–free and risky assets are denoted α and β, respectively. The investor solves the following optimization problem:

$$\begin{aligned} \mathbf{P}: \quad & \max_{\{\alpha,\beta\}} \sum_{s=1}^{S} d_s^q u(\alpha + \beta x_s) \\ & \text{s.t. } \alpha + \beta\pi^q \leq w + \pi^q. \end{aligned} \tag{8}$$

The existence of the representative investor means no–trade equilibrium occurs, that is, the demands in the equilibrium are initial wealth. Under some regularity conditions, the price of the risky asset is given as

$$\pi^q = \frac{\sum_{s=1}^{s} d_s^q x_s u'(w+x_s)}{\sum_{s=1}^{S} d_s^q u'(w+x_s)} \tag{9}$$

We demonstrate the effect of optimism (pessimism) on asset prices applying Theorem 2.1. Let us define the risk–neutral decision weight as follows:

$$\hat{d}_s^q = \frac{d_s^q u'(w+x_s)}{\sum_{s=1}^{S} d_s^q u'(x+x_s)}. \tag{10}$$

Using it, we can rewrite the equilibrium asset price as the following:

$$\begin{aligned}\pi^q &= \frac{\sum_{s=1}^{s} d_s^q x_s u'(w+x_s)}{\sum_{s=1}^{S} d_s^q u'(w+x_s)} \\ &= \sum_{s=1}^{S} \frac{d_s^q u'(w+x_s)}{\sum_{s=1}^{s} d_s^q u'(w+x_s)} x_s \\ &= \sum_{s=1}^{S} \hat{d}_s^q x_s.\end{aligned} \tag{11}$$

In a word, the equilibrium asset price is equal to the expectation of the discrete random variable $\boldsymbol{x} = (x_1, x_2, \ldots, x_S)$ with the risk–neutral decision weight $\hat{\boldsymbol{d}}^q = (\hat{d}_1^q, \hat{d}_2^q, \ldots, \hat{d}_S^q)$. We consider a more pessimistic (optimistic) PWF r than q. By Theorem 2.1, the risk–neutral decision weight $\hat{\boldsymbol{d}}^q$ dominates (is dominated by) $\hat{\boldsymbol{d}}^r$ in the sense of MLRD. Since MLRD implies FSD, we have

$$\pi^q = \sum_{s=1}^{S} \hat{d}_s^q x_s \geq (\leq) \sum_{s=1}^{S} \hat{d}_s^r x_s = \pi^r. \tag{12}$$

We summarize the discussion into the following result:

Result 3.1. More optimistic (pessimistic) representative investors lead to icreases (decreases) in equilibrium asset prices.

3.2 Comparative risk aversion under background risk

We consider a DM with the RDEU representation facing both controllable and uncontrollable risks, which are mutually independent random variables. The uncontrollable risk is usually called background risk. The RDEU is exhibited by a PWF q and an increasing utility function u. A realization of the controllable risk is x. And the background risk is represented by a discrete random variable $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_S)$ with a probability vector $\boldsymbol{p} = (p_1, p_2, \dots, p_S)$. We determine the effect of optimism (pessimism) on Arrow–Pratt (absolute) risk aversion under background risk (Arrow; 1971, Pratt; 1964).

Following Nachman (1982), we define the derived utility function:

$$v(x; q) = \sum_{s=1}^{S} d_s^q u(x + \epsilon_s). \tag{13}$$

The first– and second–order derivatives of the utility function is given as:

$$v'(x; q) = \sum_{s=1}^{S} d_s^q u'(x + \epsilon_s), \tag{14}$$

$$v''(x; q) = \sum_{s=1}^{S} d_s^q u''(x + \epsilon_s). \tag{15}$$

Then, Arrow–Pratt risk aversion under background risk is given as:

$$\mathcal{A}(x; v, q) = -\frac{v''(x; q)}{v'(x; q)} = -\frac{\sum_{s=1}^{S} d_s^q u''(x + \epsilon_s)}{\sum_{s=1}^{S} d_s^q u'(x + \epsilon_s)}. \tag{16}$$

Let us define the risk–neutral decision weight

$$\hat{d}_s^q = \frac{d_s^q u'(x + \epsilon_s)}{\sum_{s=1}^{S} d_s^q u'(x + \epsilon_s)}. \tag{17}$$

Using it, we can rewrite Arrow–Pratt risk aversion under background risk as the following:

$$\begin{aligned} \mathcal{A}(x; v, q) &= -\frac{\sum_{s=1}^{S} d_s^q u''(x + \epsilon_s)}{\sum_{s=1}^{S} d_s^q u'(x + \epsilon_s)} \\ &= \sum_{s=1}^{S} \frac{d_s^q u'(w + \epsilon_s)}{\sum_{s=1}^{S} d_s^q u'(w + \epsilon_s)} \left(-\frac{u''(x + \epsilon_s)}{u'(x + \epsilon_s)} \right) \\ &= \sum_{s=1}^{S} \hat{d}_s^q \mathcal{A}(x + \epsilon_s; u). \end{aligned} \tag{18}$$

In a word, Arrow–Pratt risk aversion under background risk is equal to the expectation of the discrete random variable $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_S)$ with the risk–neutral decision weight $\hat{\boldsymbol{d}}^q = (\hat{d}_1^q, \hat{d}_2^q, \ldots, \hat{d}_S^q)$. We assume that the utility function exhibits decreasing absolute risk aversion and consider a more pessimistic (optimistic) PWF r than q. By the similar discussion of the previous subsection, we have that

$$\mathcal{A}(x; v, q) = \sum_{s=1}^{S} \hat{d}_s^q \mathcal{A}(x + \epsilon_s) \leq (\geq) \sum_{s=1}^{S} \hat{d}_s^r \mathcal{A}(x + \epsilon_s) = \mathcal{A}(x; v, r). \tag{19}$$

We summarize the discussion into the following result:

Result 3.2. Suppose that utility functions display decreasing absolute risk aversion. More optimistic (pessimistic) decision makers behave in a more risk–tolerant (risk–averse) manner under background risk.

4 Conclusion

Optimism and pessimism influence most decision–makings under risk. Rank dependent expected utility theory can express them by the shape of probability weighting functions. Convex and concave shift of the probability weighting function capture them. It is not enough to express optimism and pessimism, but it is also important to understand how they influence economic problems under risk. This is a motivation of our new characterization of optimism and pessimism.

The characterization of optimism and pessimism is given as monotone likelihood ratio dominance between their corresponding risk–neutral decision weights. This characterization has sharp comparative static predictions for economic problems under risk parallel to the expected utility theory. We display the following two comparative static predictions. First, more optimistic (pessimistic) representative investors lead to increases (decreases) in asset prices, Second more optimistic (pessimistic) decision makers behave in a more risk–bear (risk–averse) manner under background risk if they exhibit decreasing absolute risk aversion.

Finally, we give a comment on future research. Experimental evidences suggest inverse S–shape probability weighting functions. However, our characterization does not predict how this shape influence decision problems under risk compared to the expected utility theory. This research remains to be completed.

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