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Comparative Risk Aversion for State-Dependent Preferences

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Abstract

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1 Introduction

The idea that preferences over uncertain outcomes may depend on the state of nature in which a given outcome is realised seems intuitively appealing. Common examples include states of nature associated with death, injury, illness or accident. Karni (1985) formalized this idea, presenting a state-dependent version of the expected utility model. Karni (1987) extended the analysis to generalized expected utility theory.

A crucial issue in any analysis of preferences concerning uncertainty is the concept of risk-aversion. Important aspects of this issue include the definition of risk-aversion, comparisons of the riskiness of alternative state-contingent consumption bundles, comparisons of risk-aversion between individuals and comparisons of risk aversion for a given individual at different wealth levels. In particular, the hypothesis of decreasing absolute risk aversion (DARA) has played a central role in comparative static analysis.

Karni (1983,1985, 1987) analyzes a number of these issues. His central idea that the set of riskless outcomes may be replaced by a more general reference set, representing the most desirable allocation of possible wealth levels at actuarially fair prices. Karni develops a notion of DARA for state-contingent preferences. However, this notion is relevant to autocomparable preferences (those characterized by affine reference sets). and, Except for the case of additively separable (state-dependent expected utility) preferences, it does not yield sharp comparative static results.

The idea that preferences may be state-dependent fits naturally with an analysis of uncertainty based on a representation of random variables as state-contingent consumption or production bundles analogous to consumption and production bundles in standard consumer and producer theory. Chambers and Quiggin (2000) show that the state-contingent approach may be used to characterise risk aversion using standard tools of modern consumer theory such as expenditure functions. Chambers and Quiggin (2003) develop a range of primal and dual measures of risk aversion.

This paper shows how these concepts of risk-aversion may be extended to the case of state-dependent preferences, whether or not these preferences are autocomparable. We characterize autocomparability as a special case. We show how standard comparative static results, originally derived for the state-independent expected utility model, may be extended to general state-dependent preferences, without the requirement for additive separability.

2 Notation

We consider preferences over random variables represented as mappings from a state space Ω to a convex outcome space $Y \subseteq \Re$. Our focus is on the case where Ω is a finite set $\{1, ...S\}$, and the space of random variables is $Y^S \subseteq \Re^S$. The unit vector is denoted $\mathbf{1} = (1, 1, ...1)$, and we define $\mathbf{e}_i$ as the i-th row of the $S \times S$ identity matrix

$$\mathbf{e}_i = (0, ..., 1, 0, ..., 0).$$

Probabilistic beliefs are defined by a vector $\widehat{\boldsymbol{\pi}} \in \Re_+^S$ such that $\sum_s \hat{\pi}_s = 1$. The way in which such beliefs may be elicited from individuals with state-dependent preferences, is discussed by Karni (1999) and Grant and Karni (2005).

Preferences over state-contingent incomes are given by an ordinal mapping $W : \Re^S \rightarrow \Re$. W is continuous, nondecreasing, and quasi-concave in $\mathbf{y}$. The least-as-good sets associated with this preference ordering are given by

$$V(w) = \{\mathbf{y} : W(\mathbf{y}) \geq w\}.$$

Important examples are (state-independent) expected utility preferences

$$W(\mathbf{y}) = \sum_s \hat{\pi}_s u(y_s) \tag{1}$$

(where $u : \Re \rightarrow \Re$ is assumed concave), and state-dependent expected utility preferences

$$W(\mathbf{y}) = \sum_s \hat{\pi}_s u^s(y_s) \tag{2}$$

(where each $u^s : \Re \rightarrow \Re$ is assumed concave).

For any vector of state-contingent prices, $\mathbf{p} \in \Re_{++}^S$, and given income, m, we can represent W by the indirect utility function

$$I(\mathbf{p}, m) = \max_{\mathbf{y}} \{W(\mathbf{y}) : \mathbf{p}\mathbf{y} \leq m\}.$$

The associated expenditure function is defined:

$$\begin{aligned} E(\mathbf{p}, w) &= \inf\{\mathbf{p}\mathbf{y} : \mathbf{y} \in V(w)\} \\ &= \inf\{m : I(\mathbf{p}, m) \geq w\}. \end{aligned}$$

Define

$$\begin{aligned}\hat{\mathbf{y}}(\mathbf{p},w) &= \arg\min\{\mathbf{p}'\mathbf{y} : W(\mathbf{y}) \geq w\} \\ &= \partial E(\mathbf{p},w).\end{aligned}$$

Here ∂ denotes the superdifferential with respect to $\mathbf{p}$. Thus, $\hat{\mathbf{y}}(\mathbf{p},w)$ is the set of state-contingent income vectors that would minimise the cost of achieving welfare level w, given state-contingent prices $\mathbf{p}$. For strictly quasiconcave W, $\hat{\mathbf{y}}(\mathbf{p},w)$ is a singleton.

3 Reference sets and risk premiums

Karni (1985) defines the *reference set* as "...the optimal distribution of wealth across states of nature that is chosen by a risk-averse decision maker facing fair insurance" at the objective probabilities $\widehat{\boldsymbol{\pi}}$. For income m, this distribution is given by $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},I(\mathbf{p},m))$.[1] For given $\mathbf{y}$, the element (or, more generally subset) of the reference set yielding welfare $W(\mathbf{y})$ is the reference equivalent $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},W(\mathbf{y}))$. Thus the expenditure on the reference equivalent

$$\hat{\boldsymbol{\pi}}'\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},W(\mathbf{y})) = E(\hat{\boldsymbol{\pi}},\mathbf{W}(\mathbf{y}))$$

may be interpreted as the reference-equivalent income (or minimal-equivalent income).

The reference set corresponds in a consumer context to the consumer's income-expansion path given $\hat{\boldsymbol{\pi}}$. For given $\hat{\boldsymbol{\pi}}$, the reference set, $\hat{Y}(\hat{\boldsymbol{\pi}}) \subset \Re^S$, is thus

$$\hat{Y}(\hat{\boldsymbol{\pi}}) = \cup_w \{\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},w)\}.$$

For strictly risk-averse, probabilistically sophisticated, state-independent preferences with subjective probabilities $\hat{\boldsymbol{\pi}}$, $\hat{Y}(\hat{\boldsymbol{\pi}})$, the reference set for the price vector $\mathbf{p} = \hat{\boldsymbol{\pi}}$, is simply the certainty ray $\{c\mathbf{1} : c \in \Re\}$. This can easily be checked for the case of expected-utility preferences (1). By contrast, for state-dependent expected-utility preferences (2), the reference set will not coincide with the certainty ray when the price vector $\mathbf{p}$ coincides with the decision maker's beliefs $\hat{\boldsymbol{\pi}}$, unless all the utility functions u^s are identical (up to an additive shift).

[1] $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},m)$ need not be unique. In particular, for risk-neutral preferences, any $\mathbf{y}$ will be an element of the reference set for $m = \hat{\boldsymbol{\pi}}'\mathbf{y}$. However, we will focus attention on the case of strictly risk-averse preferences, where $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},m)$ is unique.

3.1 Risk premiums

Following the standard analysis of the state-independent and state-dependent cases, we define absolute and relative risk premiums

$$\begin{aligned} a(\mathbf{y};\hat{\boldsymbol{\pi}}) &= \hat{\boldsymbol{\pi}}'\mathbf{y} - E(\hat{\boldsymbol{\pi}}, W(\mathbf{y})) \\ r(\mathbf{y};\hat{\boldsymbol{\pi}}) &= \frac{\hat{\boldsymbol{\pi}}'\mathbf{y}}{E(\hat{\boldsymbol{\pi}}, W(\mathbf{y}))} \end{aligned}$$

The absolute risk premium is the difference between the value of $\mathbf{y}$ using $\hat{\boldsymbol{\pi}}$ and the reference-equivalent income, that is minimal expenditure required to reach the same level of preference as $\mathbf{y}$ at prices $\hat{\boldsymbol{\pi}}$. It is evident that $a(\mathbf{y}, \hat{\boldsymbol{\pi}})$ is the Hicksian equivalent variation for a shift to $\mathbf{y}$ from $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}}, W(\mathbf{y}))$ and that the literature on consumer surplus and other approximations to the compensating and equivalent variations can be applied to yield close approximations for $a(\mathbf{y}, \hat{\boldsymbol{\pi}})$ and related measures (e.g., Diewert, 1992).

We can express the risk premium as the difference between two expenditure functions. Let $y^* \in Y(\hat{\boldsymbol{\pi}})$ and $\boldsymbol{\pi}'\mathbf{y}^* = \boldsymbol{\pi}'\mathbf{y}$, then

$$\begin{aligned} a(\mathbf{y};\hat{\boldsymbol{\pi}}) &= E(\hat{\boldsymbol{\pi}}, W(\mathbf{y}^*)) - E(\hat{\boldsymbol{\pi}}, W(\mathbf{y})) \\ &= E(\hat{\boldsymbol{\pi}}, W(\mathbf{y} + (\mathbf{y}^* - \mathbf{y}))) - E(\hat{\boldsymbol{\pi}}, W(\mathbf{y})), \end{aligned}$$

so that we can think of the risk premium as the willingness to pay to avoid the actuarially fair risk $(\mathbf{y}^* - \mathbf{y})$.

Observe that by the definition of the expenditure function

$$\hat{\boldsymbol{\pi}}'\mathbf{y} \geq E(\hat{\boldsymbol{\pi}}, W(\mathbf{y})),$$

so that $a(\mathbf{y};\hat{\boldsymbol{\pi}}) \geq 0$ with equality if and only if $\mathbf{y} \in Y(\hat{\boldsymbol{\pi}})$.

Similar interpretations are available for the relative risk premium, and we may derive $r(\mathbf{y};\hat{\boldsymbol{\pi}}) \geq 1$ with equality only for $\mathbf{y} \in Y(\hat{\boldsymbol{\pi}})$.

4 Comparisons of risk and risk aversion

4.1 Risk orderings for equal mean sets

In state-independent utility, a variety of risk orderings, $\preceq$, are used, where $\mathbf{y} \preceq \mathbf{y}'$ corresponds to various interpretations of the statement '$\mathbf{y}$ is less risky than $\mathbf{y}'$'. In all such orderings, the least risky state-contingent vectors are non-stochastic vectors of the form $c\mathbf{1}$. Risk-averse preferences are then characterized by the requirement that $W(\mathbf{y}) \leq W(\mu\mathbf{1})$ where $\mu = \hat{\boldsymbol{\pi}}'\mathbf{y}$. Most risk orderings relate only variables with the same mean, and are translation-invariant in the sense that, for all $\delta, \mathbf{y}, \mathbf{y}'$:

$$\mathbf{y} \preceq \mathbf{y}' \Leftrightarrow \mathbf{y}{+}\delta\ \mathbf{1} \preceq \mathbf{y}'{+}\delta\mathbf{1}$$

Thus, riskiness may be seen as a property of deviations from certainty, of the general form

$$\varepsilon = \mathbf{y}{-}\left(\hat{\boldsymbol{\pi}}'\mathbf{y}\right)\mathbf{1}$$

Hence, given a risk-ordering $\preceq$ we derive the induced risk ordering $\preceq^*$ on $M = \{\boldsymbol{\varepsilon} : \hat{\boldsymbol{\pi}}'\boldsymbol{\varepsilon} = 0\}$ such that

$$\boldsymbol{\varepsilon} \preceq^* \boldsymbol{\varepsilon}' \Leftrightarrow \boldsymbol{\varepsilon} + \mu\mathbf{1} \preceq \boldsymbol{\varepsilon}'{+}\mu\mathbf{1}, \quad \forall\mu$$

It is useful to apply this interpretation to specific risk orderings used in the literature. We follow the notation of Quiggin and Chambers (2004). Consider the following examples.

Example 1 *The minimal risk ordering consistent with risk aversion, requiring that receipt of mean income with certainty is preferred to the corresponding risky state-contingent income vector is denoted* $\preceq_0$. *The only risk-ordering relationships implied by* $\preceq_0$ *are of the form* $\mu(\mathbf{y})\mathbf{1} \preceq_0 \mathbf{y}$. *This ordering is translation invariant, and induces the ordering* $\preceq_0^*$ *on* M

$$\mathbf{0} \preceq_0^* \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in M.$$

Example 2 *Consider the multiplicative-spread risk ordering,* $\preceq_1$,*described by*

$$\lambda\mathbf{y} + (1-\lambda)\left(\hat{\boldsymbol{\pi}}'\mathbf{y}\right)\mathbf{1} \preceq_1 \mathbf{y}, \quad 0 \leq \lambda \leq 1$$

where $0 \leq \lambda \leq 1$. *This ordering is translation invariant, and corresponds to the requirement*

$$\lambda\boldsymbol{\varepsilon} \preceq_1^* \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \in M, \quad 0 \leq \lambda \leq 1.$$

Example 3 *The monotone spread ordering is given by*

$$\mathbf{y} \preceq_m \mathbf{y} + \boldsymbol{\varepsilon}$$

where $\hat{\boldsymbol{\pi}}'\boldsymbol{\varepsilon} = 0$ *and* $\boldsymbol{\varepsilon}$ *is comonotonic with* $\mathbf{y}$, *that is, for any* s, t

$$(y_s - y_t)(\varepsilon_s - \varepsilon_t) \geq 0$$

This ordering is translation invariant, and induces the ordering $\preceq_m^*$ on M

$$\boldsymbol{\varepsilon} \preceq_m \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}'$$

for $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$ comonotonic.

4.2 General risk orderings

To extend this analysis to the case of state-dependent preferences, we replace the certainty ray with $Y(\hat{\boldsymbol{\pi}})$ and define

$$M(\mathbf{y}, \hat{\boldsymbol{\pi}}) = Y(\hat{\boldsymbol{\pi}}) \cap \{\tilde{\mathbf{y}} : \hat{\boldsymbol{\pi}}'(\tilde{\mathbf{y}} - \mathbf{y}) = 0\}$$

as the set of points on the reference set that has the same mean as $\mathbf{y}$. When the reference set is a one-dimensional manifold, $M(\mathbf{y}, \hat{\boldsymbol{\pi}})$ is a singleton.

Now we can apply risk orderings on the basis of deviations of the form

$$\boldsymbol{\varepsilon} = \mathbf{y} - M(\mathbf{y}, \hat{\boldsymbol{\pi}}). \tag{3}$$

yielding

$$\begin{aligned} M(\mathbf{y}, \hat{\boldsymbol{\pi}}) &\preceq {}_0\mathbf{y}, \\ \lambda\mathbf{y} + (1-\lambda)M(\mathbf{y}, \hat{\boldsymbol{\pi}}) &\preceq {}_1\mathbf{y}, \quad 0 \le \lambda \le 1, \\ \mathbf{y} &\preceq {}_m\mathbf{y} + \tilde{\boldsymbol{\varepsilon}} \end{aligned}$$

where $\hat{\boldsymbol{\pi}}'\tilde{\boldsymbol{\varepsilon}} = 0$ and $\tilde{\boldsymbol{\varepsilon}}$ is comonotonic with some element of $\mathbf{y} - M(\mathbf{y}, \hat{\boldsymbol{\pi}})$. Intuitively, the definition of $\preceq_0$ says that points on the reference set are less risky than points off it with the same mean. The definition of $\preceq_1$ says that riskiness increases as we move along any line segment from $M(\mathbf{y}, \hat{\boldsymbol{\pi}})$ to $\mathbf{y}$ are less risky. The comonotone order has the same properties as in the standard case.

With this construction, it is straightforward to relax the requirement that risk orderings should relate only variables with the same mean. Observe that if $w = I(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\pi}}'\mathbf{y})$, we have

$$\mathbf{y} = \hat{\mathbf{y}}(\hat{\boldsymbol{\pi}}, w) + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon}$ is as in (3). Similarly denote

$$\mathbf{y}' = \hat{\mathbf{y}}(\hat{\boldsymbol{\pi}}, w') + \boldsymbol{\varepsilon}'$$

and any of the orderings $\preceq$ discussed above can be extended to a partial order on Y^S defined by

$$\mathbf{y} \preceq \mathbf{y}' \Leftrightarrow \boldsymbol{\varepsilon} \preceq \boldsymbol{\varepsilon}'$$

In the remainder of this paper, we will use $\preceq_0, \preceq_1$and $\preceq_m$to denote these more general orderings.

4.3 Risk aversion

An important reason for defining a generalized risk premium is to permit the comparison of risk aversion across individuals. As Karni (1985) observes, comparisons of risk aversion are feasible for individuals who share common beliefs $\hat{\boldsymbol{\pi}}$ and a reference set $Y(\hat{\boldsymbol{\pi}})$. Comparing two individuals i and j who have common beliefs and a common reference set, we say that i is (absolutely) more risk-averse than j if for all $\mathbf{y}$

$$a^i(\mathbf{y}; \hat{\boldsymbol{\pi}}) \geq a^j(\mathbf{y}; \hat{\boldsymbol{\pi}}). \tag{4}$$

Departures from the objective probabilities incur greater costs in maintaining the reference utility for the more risk-averse person. Equivalently i is more risk averse than j if for all $\mathbf{y}$

$$\hat{\boldsymbol{\pi}}'\mathbf{y} - E^i\left(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y})\right) \geq \hat{\boldsymbol{\pi}}'\mathbf{y} - E^j\left(\hat{\boldsymbol{\pi}}, W^j(\mathbf{y})\right),$$

or

$$E^j\left(\hat{\boldsymbol{\pi}}, W^j(\mathbf{y})\right) \geq E^i\left(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y})\right).$$

This also implies a notion of relatively more risk averse by a parallel definition. More significantly, it implies that more risk averse people are characterized by the requirement that given state-claim prices corresponding to their probabilities, they would always spend less on attaining the level of utility offered by $\mathbf{y}$ than less risk averse individuals.

It may seem counterintuitive that more risk-averse individuals should spend less on their reference equivalent than less risk-averse individual. However, this has an exact parallel in state-independent expected utility theory in more risk-averse individuals having lower certainty equivalents than less risk-averse individuals. Given that the expansion path is monotonically increasing in w, more risk-averse individuals always find their reference-equivalent consumption bundle "closer to the origin" in the reference set than less risk-averse individuals.

More general comparisons of risk aversion are also useful. Suppose $\preceq$ is a risk ordering for both i and j.Then, we say that i is more risk-averse than j for $\preceq$ if , for all $\mathbf{y}, \mathbf{y}'$

$$\mathbf{y} \preceq \mathbf{y}' \Rightarrow a^i(\mathbf{y}'; \hat{\boldsymbol{\pi}}) - a^i(\mathbf{y}; \hat{\boldsymbol{\pi}}) \geq a^j(\mathbf{y}'; \hat{\boldsymbol{\pi}}) - a^j(\mathbf{y}; \hat{\boldsymbol{\pi}}) \tag{5}$$

or equivalently

$$E^i\left(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y})\right) - E^i\left(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y}')\right) \geq E^j\left(\hat{\boldsymbol{\pi}}, W^j(\mathbf{y})\right) - E^j\left(\hat{\boldsymbol{\pi}}, W^j(\mathbf{y}')\right).$$

This paragraph is still garbled. I think you sent me the wrong version, but just in case, I will try to be a little clearer about what I am objecting to. There is notation that is not defined. We do not have a definition of $\preceq_Y$ or $\preceq_{\mathcal{P}}$. What do they order and where do they come from. I also question the notational wisdom of using $\preceq_I$ to denote the ordering on I because up until this point, this notation has referred to risk orderings. Regardless, everything needs to be defined and clearly so people can understand what's going on. This definition may be restated in the terminology of supermodularity theory (Topkis, 1998). Consider a family I of individuals with common beliefs $\hat{\boldsymbol{\pi}}$ and a common expansion path $Y(\hat{\boldsymbol{\pi}})$and an ordering $\preceq_I$ on I. If (5) holds whenever $j \preceq_I i$, then the risk premium $a^i(\mathbf{y}; \hat{\boldsymbol{\pi}})$ and the expenditure function $E^i(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y}))$ display increasing differences in $i, \mathbf{y}$ relative to the orderings $\preceq_Y$ and $\preceq_I$ Equivalently an ordering $\preceq_I$ on I is an ordering of risk aversion, given $\preceq_{\mathcal{P}}$ if $a^i(\mathbf{y}; \hat{\boldsymbol{\pi}})$ and $E^i(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y}))$ are supermodular in i and $\mathbf{y}$, given the orderings $\preceq_Y$ and $\preceq_I$. Note that the difference between $a^i(\mathbf{y}; \hat{\boldsymbol{\pi}})$ and $E^i(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y}))$ is given by $\hat{\boldsymbol{\pi}}'\mathbf{y}$ which is a valuation on $\mathbf{y}$, so that supermodularity of $a^i(\mathbf{y}; \hat{\boldsymbol{\pi}})$ is equivalent to supermodularity of $E^i(\hat{\boldsymbol{\pi}}, W^i(\mathbf{y}))$

The general definition in terms of supermodularity theory reduces to the basic definition (4) if we consider the ordering $\preceq_0$ defined by the sole requirement $\mathbf{y}(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\pi}}\mathbf{y}) \preceq \mathbf{y}$, $\forall \mathbf{y}$. **$\mathbf{y}(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\pi}}\mathbf{y})$ is leftover from an old version, we need to replace it.**

5 Constant, decreasing and increasing risk aversion

Concepts of constant, decreasing and increasing risk aversion play a crucial role in the literature on problems of economic choice under uncertainty that has been developed for state-independent models of preferences. If models of state-dependent preferences are to be applied to such problems, it is important to consider the extent to which such concepts can be generalized. We will focus on absolute risk aversion. The extension to concepts of constant, decreasing and increasing relative risk aversion is straightforward if the reference set is a ray from the origin, but more complex in other cases.

5.1 Constant risk aversion and linear risk tolerance

With state-independent preferences, risk-averse for probabilities $\hat{\boldsymbol{\pi}}$, the reference set for $\hat{\boldsymbol{\pi}}$ is the bisector or certainty ray $c\mathbf{1}$. Under constant absolute risk aversion (CARA), for any $\mathbf{p}$, the expansion path is parallel to $c\mathbf{1}$. It follows that, in any standard[2] choice problem, an increase in wealth of δ simply produces a shift of $\delta\mathbf{1}$ in the optimal consumption vector. Similarly, under constant relative risk aversion (CRRA), the expansion path is a ray from the origin. This leads us to the following definition of these concepts in terms of reference sets.

Definition 4 *Preferences satisfy constant absolute risk aversionif for all* $\hat{\boldsymbol{\pi}}$, $Y(\hat{\boldsymbol{\pi}}) + \beta\mathbf{1} \subseteq Y(\hat{\boldsymbol{\pi}})$, $\beta \in \Re$. *Preference satisfy constant relative risk aversion if for all* $\hat{\boldsymbol{\pi}}$, $\mu Y(\hat{\boldsymbol{\pi}}) \subseteq Y(\hat{\boldsymbol{\pi}})$, $\mu > 0$.

Ordinal preferences, therefore, satisfy CARA if there exists a utility normalization such that, for all $\boldsymbol{\pi}$,

$$\begin{aligned} E(\boldsymbol{\pi}, w) &= E(\boldsymbol{\pi}, 0) + w\boldsymbol{\pi}'\mathbf{1} \\ &= E(\boldsymbol{\pi}, 0) + w \end{aligned}$$

under the normalization that $\boldsymbol{\pi}'\mathbf{1} = 1$ (which is harmless for probabilities). Hence,

$$Y(\hat{\boldsymbol{\pi}}) = \partial E(\hat{\boldsymbol{\pi}}, 0) + \cup_w \{w\mathbf{1}\}$$

The primal implication is that

$$V(w) = V(0) + w\mathbf{1},$$

which is equivalent to the definition in terms of reference sets.

[2] A standard choice problem is one involving allocation of wealth across a range of assets or other activiities, such that changes in wealth shift the choice set parallel to $c\mathbf{1}$.

Similarly, ordinal preferences satisfy CRRA if there exists a utility normalization $(w > 0)$ such that for all $\boldsymbol{\pi}$,

$$E(\boldsymbol{\pi},w) = wE(\boldsymbol{\pi},1),$$

so that

$$Y(\hat{\boldsymbol{\pi}}) = \cup_w \{w\partial E(\hat{\boldsymbol{\pi}},1)\}$$

The primal implication is that

$$V(w) = wV(1).$$

We, therefore, generalize the notion of CARA as follows

Definition Preferences display nonlinear CARA if

$$V(w) = V(0) + g(w),$$

with $g : \Re \to \Re^S$ and $g(0) = \mathbf{0}$.

Thus, preferences display nonlinear CARA if and only if

$$E(\boldsymbol{\pi},w) = E(\boldsymbol{\pi},0) + \pi' g(w),$$

so that preferences are risk averse with respect to $\hat{\boldsymbol{\pi}}$ if and only if

$$\hat{Y}(\hat{\boldsymbol{\pi}}) = \partial E(\hat{\boldsymbol{\pi}},0) + \cup_w \{g(w)\}.$$

Under this definition expansion paths are 'parallel' to the manifold $\cup_w \{g(w)\}$, which allows for nonlinear responsiveness to real wealth (w) changes.

That is, all income effects on state-contingent incomes are independent (in a direct sense) of state-contingent prices. The income effects are measured by $g(w)$ and by the rate at which w changes with income. Let the vector of partial derivatives of $g(w)$ with respect to w be denoted $g'(w) \in \Re^S$. Then, presuming differentiability, the income effects are measured by

$$\begin{aligned} \frac{\partial}{\partial m} g(I(\boldsymbol{\pi},m)) &= g'(I(\boldsymbol{\pi},m)) I_m(\boldsymbol{\pi},m) \\ &= \frac{g'(I(\boldsymbol{\pi},m))}{E_w(\boldsymbol{\pi}, I(\boldsymbol{\pi},m))} \end{aligned}$$

This leads to an absolute risk premium of the form

$$\begin{aligned} a(\mathbf{y};\hat{\boldsymbol{\pi}}) &= \hat{\boldsymbol{\pi}}'\mathbf{y} - E(\hat{\boldsymbol{\pi}},W(\mathbf{y})) \\ &= \hat{\boldsymbol{\pi}}'\mathbf{y} - E(\hat{\boldsymbol{\pi}},0) - \hat{\boldsymbol{\pi}}'g(W(\mathbf{y})). \end{aligned}$$

which implies:

Proposition 5 *Preferences display nonlinear CARA if and only if the absolute risk premium $a(\mathbf{y};\hat{\boldsymbol{\pi}})$ is constant for shifts parallel to the reference set. That is, for any w and $\mathbf{y} \in V(0)$*

$$a(\mathbf{y}+g(w);\hat{\boldsymbol{\pi}}) = a(\mathbf{y};\hat{\boldsymbol{\pi}})$$

P roof. *For $\mathbf{y} \in V(0)$*

$$a(\mathbf{y};\hat{\boldsymbol{\pi}}) = \hat{\boldsymbol{\pi}}'\mathbf{y} - E(\hat{\boldsymbol{\pi}},0).$$

Under nonlinear CARA for $\mathbf{y} \in V(0)$, $\mathbf{y} + g(w) \in V(w)$, whence

$$\begin{aligned} a(\mathbf{y}+g(w);\hat{\boldsymbol{\pi}}) &= \hat{\boldsymbol{\pi}}'\mathbf{y} + \hat{\boldsymbol{\pi}}'g(w) - E(\hat{\boldsymbol{\pi}},0) - \hat{\boldsymbol{\pi}}'g(w) \\ &= \hat{\boldsymbol{\pi}}'\mathbf{y} - E(\hat{\boldsymbol{\pi}},0) \\ &= a(\mathbf{y};\hat{\boldsymbol{\pi}}). \end{aligned}$$

■

Another especially convenient class of preferences are the linear risk tolerant preferences, which correspond in the standard consumer case to the class of quasi-homothetic preferences, whose demands assume the Gorman Polar form. We have:

Definition 6 *Preferences satisfy linear risk tolerance if*

$$V(w) = V^0 + wV^1,$$

where $V^0, V^1 \subset Y^S$.

The importance of the linear risk tolerant (LRT) class for state-dependent preferences emerges in attempts to generalize the Pratt-Arrow notions of decreasing and increasing absolute risk aversion for state-dependent preferences. In his generalizations of these concepts, Karni (1985) restricts attention to *autocomparable* preferences. Preferences are *autocomparable* if the reference set is affine.[3]

To admit the possibility of multiple solutions to the expected-value problem, we have:

Definition 7 *An individual's preferences are autocomparable for* $\hat{\boldsymbol{\pi}}$ *if* $Y(\hat{\boldsymbol{\pi}}) = Y^0(\hat{\boldsymbol{\pi}}) + Y^1(\hat{\boldsymbol{\pi}})$ *with* $Y^0(\hat{\boldsymbol{\pi}}) = \partial E(\hat{\boldsymbol{\pi}},0) \subset Y^S$ *and* $Y^1(\hat{\boldsymbol{\pi}})$ *a cone, i.e.,* $\mu Y^1(\hat{\boldsymbol{\pi}}) \subseteq Y^1(\hat{\boldsymbol{\pi}}), \mu > 0$.

Preferences can be autocomparable for some $\hat{\boldsymbol{\pi}}$ but not for others. For example, state-independent expected utility preferences are autocomparable for the subjective probabilities that parametrize the expected-utility function, but not necessarily for other probabilities. It is trivial from what has gone before that CARA, CRRA, and LRT preferences are all autocomparable *for all possible probability distributions.* Generalized CARA preferences are only autocomparable if the manifold $\cup_w \{g(w)\}$ is a cone, but in this case, such preferences can always be renormalzied to be CARA preferences.

5.2 Decreasing and increasing absolute risk aversion

Quiggin and Chambers (2004) show that, like other notions of comparative risk aversion, notions of decreasing and increasing absolute and relative risk aversion are most compactly and usefully expressed in the language of supermodularity theory. With the notions of risk ordering and the reference set, developed above, it is straightforward to extend the definition of decreasing absolute risk aversion (DARA) presented by Quiggin and Chambers (2004) for risk-averse state-independent preferences to the case of general reference sets.

Definition 8 *Preferences display DARA with respect to a risk ordering* $\preceq^*$ *on E, for a given* π, *on* Y *if* $a(\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},w) + \boldsymbol{\varepsilon}; \hat{\boldsymbol{\pi}})$ *is submodular in* w *and* $\boldsymbol{\varepsilon}$

The discussion of autocomparability presented above suggests an alternative approach, closer to that adopted by Karni. Suppose that the reference

[3] Karni (1985) uses the term linear, but his definition is equivalent to requiring the elements of a particular point in the reference set to be affine translates of one another.

set for $\hat{\boldsymbol{\pi}}$ takes the affine form

$$Y(\hat{\boldsymbol{\pi}}) = Y^0(\hat{\boldsymbol{\pi}}) + Y^1(\hat{\boldsymbol{\pi}})$$

Under CARA, this will be true for all $\boldsymbol{\pi}$, so Y^1 is independent of π. Hence, for $\mathbf{y}^1 \in Y^1(\hat{\boldsymbol{\pi}})$,

$$V\left(W\left(\mathbf{y}+c\mathbf{y}^1\right)\right) = V(W(\mathbf{y}))+c\mathbf{y}^1$$

for all $\mathbf{y}$,c and in particular all $\hat{\mathbf{y}} \in \hat{Y}(\hat{\boldsymbol{\pi}})$.

With decreasing risk-aversion, the addition of consumption $c\ \mathbf{y}^1(\hat{\boldsymbol{\pi}})$ should make the individual more willing to accept movement away from the reference set, implying that, for all $\hat{\mathbf{y}} \in \hat{Y}(\hat{\boldsymbol{\pi}})$

$$V(W(\hat{\mathbf{y}}))+c\mathbf{y}^1(\hat{\boldsymbol{\pi}}) \subseteq V\left(W\left(\hat{\mathbf{y}}+c\mathbf{y}^1(\hat{\boldsymbol{\pi}})\right)\right) \tag{6}$$

This kind of comparison is feasible only for autocomparable preferences.

These two approaches can be related using the following proposition, which shows that the definition based on autocomparable preferences is a special case of the supermodularity definition.

Proposition 9 *Suppose* ***There is no such reference in this version of the paper??*** *is satisfied. Then preferences satisfy (6) if and only if* $a\left(\mathbf{y}+c\mathbf{y}^1(\hat{\boldsymbol{\pi}});\hat{\boldsymbol{\pi}}\right)$ *is submodular in* c *and* $\mathbf{y}$ *with respect to the ordering* $\preceq_0$ *.*

Proof: Since the ordering $\preceq_0$ requires only $\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\pi}}\mathbf{y}) \preceq_0 \mathbf{y}$, and $a(\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\pi}}\mathbf{y});\hat{\boldsymbol{\pi}}) = 0$, $a\left(\mathbf{y}+m\mathbf{y}^1(\hat{\boldsymbol{\pi}});\hat{\boldsymbol{\pi}}\right)$ is submodular in c and $\mathbf{y}$ with respect to the ordering $\preceq_0$ if and only if $a\left(\mathbf{y}+c\mathbf{y}^1(\hat{\boldsymbol{\pi}});\hat{\boldsymbol{\pi}}\right)$ is decreasing in c which will be true if and only if (6) holds.

6 Implications for asset demand

We can now extend the results for the standard two-asset portfolio demand problem to the case of preferences with an affine expansion set for π. Consider an individual with reference set

$$\hat{\mathbf{y}}(\hat{\boldsymbol{\pi}},w) = \left\{\mathbf{y}(\hat{\boldsymbol{\pi}},0) + w\mathbf{y}^1(\hat{\boldsymbol{\pi}})\right\} \tag{7}$$

stochastic endowment $\mathbf{e}$ and wealth w **We can't use w as wealth. We used it as welfare. Need to change it in what follows.**that can be allocated between two assets, 1and 2, with payoffs $\mathbf{y}^1$ and $\mathbf{y}^2$ where $\mathbf{y}^1$ is the direction of the reference set and

$$\hat{\boldsymbol{\pi}}\mathbf{y}^2 = \hat{\boldsymbol{\pi}}\mathbf{y}^1 = 1$$

We will treat asset 1 as numeraire with price equal to 1, and denote the price of asset 2 by $p_2 < 1$. Thus, $\mathbf{y}^1$ and $\mathbf{y}^2$ correspond respectively to the safe asset and the risky asset in the standard analysis. Let α denote purchases of the risky asset.

Thus the optimisation problem is

$$\max_a W\left(\mathbf{e}+\frac{(w-\alpha)}{p_2}\mathbf{y}^1 + \alpha\mathbf{y}^2\right)$$

We will assume that the base allocation $\mathbf{y}^0$ is in the span of the market and more precisely that there exist b^1, b^2 such that

$$\mathbf{y}^0 = \mathbf{e}+b_1\mathbf{y}^1 + b_2\mathbf{y}^2$$

In particular, this encompasses the special case $\mathbf{e} = \mathbf{y}^0, b_1 = b_2 = 0$.

We then obtain

Proposition 10 *Assume preferences display DARA with respect to* $\preceq_1$ *. Then the following are sufficient conditions for an increase in optimal purchases of the asset: (i) an increase in wealth* w*; (ii) a reduction in the asset price* p_2*; (iii) the replacement of the return vector* $\mathbf{y}^2$ *by* $\mathbf{y}^{2\prime}$ *where* $\mathbf{y}^{2\prime} \preceq_m \mathbf{y}^2$.

Proof: Consider the initially optimal α and some $\alpha' < \alpha$. Let

$$\begin{aligned} \mathbf{y} &= \mathbf{e}+\frac{(w-\alpha)}{p_2}\mathbf{y}^1 + \alpha\mathbf{y}^2 \\ \mathbf{y}' &= \mathbf{e}+\frac{(w-\alpha')}{p_2}\mathbf{y}^1 + \alpha'\mathbf{y}^2 \end{aligned}$$

Observe that $\mathbf{y}' \preceq_1 \mathbf{y}-d\mathbf{y}^1$ where

$$d = (\alpha - \alpha')(1 - p_2)$$

The optimality of α implies

$$a(\mathbf{y}; \hat{\boldsymbol{\pi}}) - a(\mathbf{y}'; \hat{\boldsymbol{\pi}}) < \delta$$

Now consider a shift from initial wealth w to $w+\delta$. If preferences display DARA with respect to $\preceq_1$, then

$$a\left(\mathbf{y}+\delta\mathbf{y}^1;\hat{\boldsymbol{\pi}}\right)-a\left(\mathbf{y}'+\delta\mathbf{y}^1;\hat{\boldsymbol{\pi}}\right)<\delta$$

and so the optimal allocation for wealth $w+\delta$ cannot be $\alpha'<\alpha$. The other cases are proved similarly.

7 Concluding comments

Models of state-dependent preferences have a number of attractive properties. Applicability of such models has, however, been limited by the lack of analytical tools comparable to those available for state-independent preferences.

Using the state-contingent approach and the tools of modern consumer and producer theory, much of the analysis of risk aversion developed for state-independent preferences may be extended to the case of state-dependent preferences. In particular, it is possible to make *comparisons* of risk aversion with respect to a range of risk ordering.

Because state-contingent representations of problems involving uncertainty exhibit the symmetry between production and consumption familiar from analysis under certainty, a natural extension of the results derived here is to consider their applicability to problems of production under uncertainty, with the reference set being reinterpreted as an expansion path, and risk-aversion being interpreted as the existence of a cost premium for deviations from the expansion path. This issue will be addressed in future work.

7.1 References

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