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Supermodularity and Risk Aversion

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Abstract

In this paper, we consider the relationship between supermodularity and risk aversion. We show that supermodularity of the certainty equivalent implies that the certainty equivalent of any random variable is less than its mean. We also derive conditions under which supermodularity of the certainty equivalent is equivalent to aversion to mean-preserving spreads in the sense of Rothschild and Stiglitz.

1 Introduction

The concept of supermodularity has revitalized the study of comparative statics, providing a range of new tools to unify and extend existing results. Relatively little, however, seems to be known about the implications of supermodularity for preferences under uncertainty. Supermodularity of demand and cost functions has a natural economic interpretation in terms of the complementarity of goods in consumption and production. If choice under uncertainty is viewed as a problem involving bundles of state-contingent commodities or income levels, it seems natural to ask how the supermodularity relates to other properties of preferences under uncertainty, such as risk aversion. The central idea of risk aversion is that, for a given expected income, it is undesirable to have high incomes in some states of nature and low incomes in others. Put another way, for a fixed expected income, incomes are complementary across states of nature. Risk aversion, thus, seems closely associated with supermodularity of preferences over state-contingent outcomes.

The standard representation of preferences under uncertainty, the expected-utility model, appears to yield an uninteresting answer to this question. Because the expected-utility functional is an addition, that is, additively separable in its arguments, it is trivially both supermodular and submodular in state-contingent incomes or commodities.[1]

A more appropriate canonical representation of preferences under uncertainty is the certainty equivalent, considered as a real-valued function on a space of state-contingent income vectors. Quiggin and Chambers (1998) show that concepts such as constant absolute risk aversion and constant relative risk aversion can be characterized simply in terms of the certainty equivalent for general preferences, without relying on the expected-utility model. An important feature of the certainty equivalent is that it is a cardinal representation of preferences. This overcomes an objection to the use of supermodularity as a representation of complementarity, namely that supermodularity is not invariant under monotone transformations.

In this paper, we consider the relationship between supermodularity and risk aversion.

[1]Some results can be obtained with respect to secondary concepts in the expected-utility model. For example, Athey (2002) shows that, under expected utility, decreasing absolute risk aversion is equivalent to the requirement that an agent's marginal utility $u'(w+s)$ should be log-supermodular in (w, s).

We draw on results dealing with the closely related concepts of Schur concavity and order-increasing rearrangements to show that supermodularity of the certainty equivalent implies that the certainty equivalent of any random variable is less than its mean. This seems to be the most basic notion of risk aversion. We also derive conditions under which supermodularity is equivalent to Schur concavity of the certainty equivalent, that is, to aversion to mean-preserving spreads in the sense of Rothschild and Stiglitz (1970). These latter conditions encompass expected-utility theory, mean-variance preferences, Yaari's (1987) dual model and the rank-dependent model of Quiggin (1982). We then illustrate how supermodularity concepts can be applied with two simple examples. Finally, we offer some concluding comments.

2 Notation

Uncertainty is represented by a state space Ω. Consistent with the literature on supermodularity and Schur concavity, we focus on the case where $\Omega = \{1...S\}$ is discrete and finite.[2] We consider preferences over state-contingent income distributions $\mathbf{y} \in \Re^S$, represented by a total ordering $\preceq$. Denote by $\mathbf{1} \in \Re^S$ the unit vector, by $\mathbf{u}_i \in \Re^S$ the ith element of the standard orthonormal basis, and by $\triangle \subset \Re^S$

$$\triangle = \{\mathbf{y} : \mathbf{y} = \mu\mathbf{1}, \mu \in \Re\} .$$

Under standard assumptions of continuity and monotonicity, a canonical representation of preferences is given by the certainty equivalent

$$e(\mathbf{y}) = \inf \{e : \mathbf{y} \preceq e\mathbf{1}\} .$$

For any preference function $W : \Re^S \rightarrow \Re$ representing $\preceq$, the certainty equivalent may be defined implicitly by the relationship

$$W(e(\mathbf{y})\mathbf{1}) = W(\mathbf{y}) .$$

[2]This approach maximizes comparability with the literature on supermodularity and rearrangement increasing functions. However results for the case when Ω is an interval with Lebesgue measure or a more general measurable set may be obtained using standard limiting arguments.

The certainty equivalent $e(\mathbf{y})$ satisfies Aczél's(1990) *agreement property*

$$e(\mu\mathbf{1}) = \mu.$$

Denote by $R(\Omega)$ the set of all permutations ρ of the set Ω, and for any $\mathbf{y} \in \Re^S, \rho \in R(\Omega)$, the permuted vector $\mathbf{y}^\rho$ where

$$y^\rho_{\rho(s)} = y_s.$$

We confine attention to the case of symmetric preferences so that for any $\mathbf{y} \in \Re^S, \rho \in R(\Omega)$,

$$e(\mathbf{y}) = e(\mathbf{y}^\rho).$$

Symmetry implies that all states have probability $1/S$. The assumption of equally probable states is purely technical, since any problem of interest can be formulated in this way (Blackorby, Davidson and Donaldson 1977). Hence, we can define the mean

$$E(\mathbf{y}) = \frac{1}{S}\sum_{s=1}^{S} y_s.$$

For the analysis in this paper, it is often sufficient to confine attention to the set $R^*(\Omega) \subseteq R(\Omega)$ consisting of rotations of order $\{1...S\}$. The rotation of order k takes 1 to k, 2 to $k+1$, S to $k-1$ and, more generally, s to $(s+k-1) \bmod S$. Thus, the rotation of order 1 is the identity.

3 Supermodularity and Schur concavity

Supermodularity is defined with respect to a partial order. Let $Y \subset \Re^K$ be a set ordered by the traditional co-ordinatewise $\leq$ relation, that is:

$$\mathbf{y} \leq \mathbf{y}' \Leftrightarrow y_k \leq y'_k \quad \forall k.$$

For any vectors, $\mathbf{y}, \mathbf{y}'$, the join and the meet, respectively, for this partial ordering are defined by

$$\begin{aligned} \mathbf{y} \vee \mathbf{y}' &= (\max\{y_1, y'_1\}, ..., \max\{y_S, y'_S\}) \\ \mathbf{y} \wedge \mathbf{y}' &= (\min\{y_1, y'_1\}, ..., \min\{y_S, y'_S\}). \end{aligned}$$

Observe that $\mathbf{y} \vee \mathbf{y}'$ is the minimal element of $\Re^K$ such that $\mathbf{y}, \mathbf{y}' \leq \mathbf{y} \vee \mathbf{y}'$.

A mapping $f : \Re^S \to \Re$ is *supermodular (submodular)* [3] if for all $\mathbf{y}, \mathbf{y}'$

$$f(\mathbf{y} \vee \mathbf{y}') + f(\mathbf{y} \wedge \mathbf{y}') \geq (\leq) f(\mathbf{y}) + f(\mathbf{y}') .$$

If $-f$ is supermodular then f is submodular. Supermodularity may also be defined in terms of *increasing differences.* A function $f : \Re^2 \to \Re$ has increasing differences if, for any $t \geq t'$, $g(x) = f(x, t) - f(x, t')$ is an increasing function of x. A function $f : \Re^S \to \Re$ has increasing differences if for any s, t and $\mathbf{x}$, the function $\hat{f} : \Re^2 \to \Re$,

$$\hat{f}(\hat{x}_s, \hat{x}_t) = f(\mathbf{x}_{-s,t}, \hat{x}_s, \hat{x}_t),$$

obtained by allowing only x_s and x_t to vary from $\mathbf{x}$ [4], has increasing differences. Topkis (1998, Corollary 2.6.1) shows that a mapping $f : \Re^S \to \Re$ is supermodular if and only if it displays increasing differences.

There is a close link between supermodularity and Schur concavity. Following Marshall and Olkin (1979), $\mathbf{y}' \in \Re^S$ is said to majorize $\mathbf{y} \in \Re^S$, denoted $\mathbf{y} \preceq \mathbf{y}'$, if, for $s = (1, 2, ..., S)$

$$\begin{aligned} \sum_{k=1}^{s} y_{[k]} &\geq \sum_{k=1}^{s} y'_{[k]} \\ \sum_{k=1}^{S} y_{[k]} &= \sum_{k=1}^{S} y'_{[k]}, \end{aligned}$$

where $y_{[k]}$ is the k-th element of the increasing rearrangement of $\mathbf{y}$, that is, the result of the permutation (not necessarily unique) such that

$$y_{[1]} \leq y_{[2]} \leq ... \leq y_{[S]}.$$

A function f is *Schur-convex* at $\mathbf{y}$ if

$$\mathbf{y} \preceq \mathbf{y}' \Rightarrow f(\mathbf{y}) \leq f(\mathbf{y}')$$

for all $\mathbf{y}'$ in a neighborhood of $\mathbf{y}$. A function is Schur-convex on a set U if it is Schur-convex at $\mathbf{y}$ for all $\mathbf{y} \in U$. If a function is Schur-convex on its entire domain, then we shall refer

[3] Supermodularity is sometimes referred to as L-superadditivity, where L is mnemonic for lattice. See, for example, Marshall and Olkin (1979).

[4] More precisely $(\mathbf{x}_{-s,t}, \hat{x}_s, \hat{x}_t)$ denotes $\mathbf{x}$ with x_s replaced by $\hat{x}_s$, and x_t replaced by $\hat{x}_t$.

to it simply as Schur-convex. A function is *Schur-concave* in these senses if $-f$ is Schur-convex. A symmetric concave function is Schur-concave, and a symmetric convex function is Schur-convex (Marshall and Olkin 1979).

If $\mathbf{y}$ and $\mathbf{y}'$ are interpreted as random variables with all states equally probable, the condition $\mathbf{y} \preceq \mathbf{y}'$ is equivalent to the statement that $\mathbf{y}$ is less risky than $\mathbf{y}'$ in the sense of Rothschild and Stiglitz (1970). Hence, in the context of choice under uncertainty, Schur concavity of the certainty equivalent is the property of aversion to increases in risk in the sense of Rothschild and Stiglitz.

'Risk aversion' is commonly identified with Schur concavity. As Machina (1984) observes, however, Schur concavity coincides with the requirement $e(\mathbf{y}) \leq E(\mathbf{y})$ in the case of expected-utility preferences, but not in general. Hence it is useful to distinguish the latter weaker notion of risk aversion explicitly from the stronger, and perhaps, more familiar notion. We, therefore, say that *e is weakly risk-averse* if, for all $\mathbf{y}$,

$$e(\mathbf{y}) \leq E(\mathbf{y}).$$

Because $E(\mathbf{y})\mathbf{1} \preceq \mathbf{y}$, Schur concavity (risk aversion in the sense of Rothschild and Stiglitz) implies weak risk-aversion. The converse is not generally true.

4 Supermodularity and risk aversion

The primary goal of this paper is to elucidate the relationship between Schur concavity of preferences (risk-aversion in the sense of Rothschild and Stiglitz) and supermodularity of the certainty equivalent. There are a number of different approaches that may be taken to the analysis of this question. We consider three: direct characterizations of supermodularity and Schur concavity; analysis based on the concept of arrangement-increasingness; and analysis based on the crucial role of the certainty ray Δ in characterizing preferences under uncertainty.

4.1 Direct characterization

For important classes of preferences, there exist well-known necessary and sufficient conditions for Schur concavity. Many are summarized in Marshal and Olkin (1979). In the case of expected utility, Schur concavity is equivalent to concavity of the utility function u. In many cases, it is possible to show that similar conditions are necessary or sufficient for supermodularity of the certainty equivalent e.

We will rely on the following result due to Topkis (1998, Lemma 2.6.4):

Lemma 1 *: If $f_i(\mathbf{x})$ is increasing and supermodular on $\Re^S$ for $i = 1 \ldots k$, T_i is a convex subset of $\Re^1$ containing the range of $f_i(\mathbf{x})$ on $\Re^S$ for $i = 1 \ldots k$, and $g(t_1, t_2 \ldots t_k, \mathbf{x})$ is supermodular in $(t_1, t_2 \ldots t_k, \mathbf{x})$ on $\left(\times_{i=1}^k T_i\right) \times X$ and increasing and convex in t_i on T_i for $i = 1 \ldots k$, and for all $t_{i'}$ in $T_{i'}$ for i' in $\{1..k\} \setminus \{i\}$ and all $\mathbf{x}$ in X, then $g(f_1(\mathbf{x}), f_2(\mathbf{x}) \ldots f_k(\mathbf{x}), \mathbf{x})$ is supermodular on $\Re^S$.*

An immediate corollary is:

Corollary 2 *Increasing expected-utility preferences are Schur-concave if and only if the associated certainty equivalent is supermodular.*

Proof Let

$$f_s(\mathbf{y}) = u(y_s)$$

for increasing $u : \Re \to \Re$ and

$$g(\mathbf{z}) = u^{-1}\left(\frac{1}{S}\sum z_s\right)$$

so that

$$e(\mathbf{y}) = g(f_1(\mathbf{y}), \ldots f_S(\mathbf{y})).$$

The f_s are increasing and are trivially both submodular and supermodular. Hence, if preferences are Schur-concave (Schur-convex), u is concave (convex), g is convex (concave), and e is supermodular (submodular). For necessity, observe that if preferences are not Schur-concave, u must be strictly convex over some interval $[a, b]$. Hence, on any sub-lattice Y of $\Re^S$ such that $Y \subseteq [a, b]^S$, e is strictly submodular.■

Lemma 1 can be used to demonstrate the equivalence of Schur concavity and supermodularity for the case of rank-ordered models of the form

$$e(\mathbf{y}) = u^{-1}\left(\sum w_{r(s)} u(y_s)\right)$$

where $r(s)$ denotes the rank of y_s, with ties broken consistently with the original ordering of the y_s (Quiggin 1982; Yaari 1987). In this case, preferences are Schur-concave if and only if u is concave and $w_1 \geq w_2 \geq ... \geq w_S$. (Chew, Karni and Safra 1987). Letting

$$f_s(\mathbf{y}) = w_{r(s)} u(y_s),$$

f_s displays increasing differences in $\mathbf{y}$ if and only if $w_1 \geq w_2 \geq ... \geq w_S$. That is, an increase in any y_t, $t \neq s$ can only lower the rank of y_s and this will always increase $w_{r(s)}$ if and only if $w_1 \geq w_2 \geq ... \geq w_S$. The remainder of the argument is as in Corollary 2.

4.2 Arrangement-increasingness

Another link between Schur concavity and supermodularity is provided by the concept of *arrangement-increasingness* (Boland and Proschan 1988; Hennessy and Lapan 2003). Consider the matrix $\mathbf{Z} \in \Re^N \times \Re^M$ constructed from a set of M column vectors $\mathbf{y} \in \Re^N$. The ordering of the rows is chosen so as to coincide with the ordering of the elements of an arbitrarily chosen vector $\mathbf{y}^1$. An *elementary rearrangement* is a reordering of the elements of some $\mathbf{y}^m$, $m = 2 ... n$ to yield $\mathbf{y}'^m$ such that either

$$\mathbf{y}'^m = \mathbf{y}^m$$

or, for a given pair c, d

$$y_k'^m = \begin{cases} y_k^m & k \neq c, d \\ y_c^m & k = d \\ y_d^m & k = c. \end{cases}$$

That is, an elementary rearrangement either leaves $\mathbf{y}^m$ unchanged or interchanges the c and d elements only. Assume, without loss of generality, that $c < d$. A rearrangement is said to be 'order-increasing' if it increases the extent to which the vectors $\mathbf{y}^m$ are similarly ordered, in the sense that $\mathbf{y}'^m = \mathbf{y}^m$ if and only if $y_c^m \leq y_d^m$. That is, for all m for which the ordering of y_c^m and y_d^m does not coincide with that of y_c^1 and y_d^1 , these elements must be interchanged.

The notion of an arrangement-increasing reordering may be illuminated further by focusing on the rows, rather than the columns, of $\mathbf{Z}$. Let $\vec{\mathbf{y}}^n \in \Re^M$ denote the n-th row of the matrix $\mathbf{Z}$. An elementary arrangement-increasing reordering of the columns of $\mathbf{Z}$ is equivalent to the replacement of two rows $\vec{\mathbf{y}}^c$ and $\vec{\mathbf{y}}^d$, $c < d$, by their meet $\vec{\mathbf{y}}^c \wedge \vec{\mathbf{y}}^d$ and join $\vec{\mathbf{y}}^c \vee \vec{\mathbf{y}}^d$ respectively. For example, let

$$\mathbf{Z} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 6 \\ 3 & 5 & 4 \end{pmatrix}.$$

Then

$$\mathbf{Z}' = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 5 & 4 \end{pmatrix}$$

represents an elementary arrangement-increasing reordering applied to rows 1 and 2 of $\mathbf{Z}$. Moreover,

$$\mathbf{Z}'' = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 5 & 6 \end{pmatrix}$$

represents an elementary arrangement-increasing reordering applied to rows 2 and 3 of $\mathbf{Z}''$. Note that since the columns of $\mathbf{Z}''$ are comonotonic (have the same ordering) no further arrangement-increasing reordering is possible. Whenever $\mathbf{Z}''$ is derived from $\mathbf{Z}$ by a finite sequence of elementary arrangement-increasing reorderings, we say that $\mathbf{Z}''$ represents an arrangement-increasing reordering of $\mathbf{Z}$.

An elementary arrangement-increasing reordering of the columns of $\mathbf{Z}$ implies a majorization of the vector $\mathbf{z} \in \Re^N$ of row sums $\sum_m \vec{\mathbf{y}}^n$, defined by

$$z_n = \sum_m Z_{nm}.$$

A function $f : \Re^N \times \Re^M \rightarrow \Re$ is *arrangement-increasing* if, whenever $\mathbf{Z}'$ is an arrangement-increasing reordering of $\mathbf{Z}$, $f(\mathbf{Z}) \leq f(\mathbf{Z}')$.

These observations form the basis of the following lemma, proved by Boland and Proschan (1988):

Lemma 3 *a) Let* $f : \Re^N \times \Re^M \to \Re, g : \Re^M \to \Re$,

$$f(\mathbf{Z}) = \sum_m g(\vec{\mathbf{y}}^m).$$

Then f *is arrangement-increasing if and only if* g *is supermodular*

.(b) Let $f : \Re^N \times \Re^M \to \Re, g : \Re^N \to \Re$

$$f(\mathbf{Z}) = g\left(\sum_m \vec{\mathbf{y}}^m\right).$$

Then f *is arrangement-increasing if and only if* g *is Schur-convex.*

If the results of Boland and Proschan are to be applied to develop a link between supermodularity and weak risk-aversion, it is necessary to focus on matrices that might be used to represent paths from $\mathbf{y}$ to $E(\mathbf{y})\mathbf{1}$. The crucial tool in applying the results of Boland and Proschan is the set of vectors:

$$\{\mathbf{y}^\rho : \rho \in R^*(\Omega)\}.$$

This set has S members, each of which is a rotation of $\mathbf{y}$. If all the y_s are different, then all the members of the set $\{\mathbf{y}^\rho : \rho \in R^*(\Omega)\}$ are distinct vectors, which can be regarded as the columns of a square matrix.

We therefore consider the matrix $\underline{\mathbf{Z}}$ having as columns the vectors $\mathbf{y}^k$, with the natural ordering, so that $\mathbf{y}^1 = \mathbf{y}$. We illustrate for the case $S = 3$:

$$\underline{\mathbf{Z}} = \begin{pmatrix} y_1 & y_3 & y_2 \\ y_2 & y_1 & y_3 \\ y_3 & y_2 & y_1 \end{pmatrix}.$$

As can be seen from the illustration, the rows of $\underline{\mathbf{Z}}$ constitute a maximally disordered[5] set of S vectors, each of which contains the elements $(y_1, y_2 ... y_S)$. We next consider the

[5]The set is maximally disordered in the sense that it cannot be derived as an order-increasing rearrangement of any other set. For $S = 3$, this set is unique, but, in general, any relabelling of the elements will yield a similarly disordered set.

maximally ordered matrix

$$\bar{\mathbf{Z}} = \begin{pmatrix} y_1 & y_1 & y_1 \\ y_2 & y_2 & y_2 \\ y_3 & y_3 & y_3 \end{pmatrix}.$$

Observe that, beginning at $\underline{\mathbf{Z}}$, we can define a sequence of order-increasing rearrangements leading to $\bar{\mathbf{Z}}$. Each member of this sequence is derived by replacing a pair of rows $\mathbf{y}^i, \mathbf{y}^j$, $i < j$, with their meet $\mathbf{y}^i \wedge \mathbf{y}^j$ and join $\mathbf{y}^i \vee \mathbf{y}^j$respectively.

Denote an arbitrary member of this sequence by $\mathbf{M}$ and consider the function defined by

$$f(\mathrm{M}) = \frac{1}{S}\sum_i e(\mathbf{y}^i).$$

We have:

$$\begin{aligned} f(\underline{\mathbf{Z}}) &= \frac{Se(\mathbf{y})}{S} \\ &= e(\mathbf{y}) \\ f(\bar{\mathbf{Z}}) &= \frac{\sum y_s}{S} \\ &= E(\mathbf{y}). \end{aligned}$$

By Lemma **??**(a), f is arrangement-increasing if and only if e is supermodular. Hence, if e is supermodular, $e(\mathbf{y}) \leq E(\mathbf{y})$. This proves:

Proposition 4 *: If e is supermodular, preferences are weakly risk-averse.*

Corollary 5 *: If preferences are Schur-convex, e is not supermodular.*

4.3 Schur concavity and the role of the certainty ray

The direct characterizations derived above show that in important cases, supermodularity and Schur concavity are equivalent. This does not seem generally true. The increasing-differences characterization of supermodularity implies that, for a twice differentiable function, supermodularity is equivalent to the requirement that all cross-partial derivatives be positive. By contrast, Schur concavity is equivalent to the requirement that the matrix of cross-partial derivatives be negative definite. These two requirements are not identical.

The example of expected utility, however, suggests that imposing appropriate consistency conditions on the certainty equivalent will generate classes of preferences over which supermodularity and Schur concavity are equivalent. An important difficulty in establishing such a relationship is that Schur concavity is a local property of preferences, in the sense that, if a function is Schur-concave at each $\mathbf{y}$ in its domain, then it is globally Schur-concave. No comparable property applies for supermodularity in general, although for twice differentiable functions, supermodularity can be characterized by the requirement that cross derivatives should be everywhere non-negative. In particular, the characterization of supermodularity in terms of rearrangements imposes restrictions on function values over sets that are not contained in any small neighborhood of $\mathbf{y}$.

In the analysis of risk preferences, however, the certainty ray, $\triangle$, plays a crucial role. And, for $\mathbf{y}$ in a sufficiently small neighborhood of some $\mu\mathbf{1}$, the set $\{\mathbf{y}^{\rho} : \rho \in R(\Omega)\}$ is contained in an arbitrarily small neighborhood of $\mathbf{y}$.

The weak definition of risk aversion relies exclusively on comparisons between vectors $\mathbf{y}$ and elements of Δ. In this section, therefore we begin by characterizing the relationship between supermodularity andweak risk aversion, before going on to derive conditions under which risk aversion in the sense of Rothschild and Stiglitz (1970) is equivalent to supermodularity of the certainty equivalent.

The crucial property is that the global behavior of any function known to be a member of the class can be determined by its behavior in an arbitrarily small neighborhood of the certainty ray Δ. That is, global preferences can be determined by preferences over small gambles. More precisely, we have:

Definition 1 *: A class E of certainty equivalents is characterized by preferences over small gambles if, for $e \in E$:(a) e is Schur-concave if and only if there exists an open set U, $\Delta \subseteq U$, such that e is Schur-concave on U; and (b) e is supermodular if and only if there exists an open set U, $\Delta \subseteq U$, such that e is supermodular on any sublattice of $\Re^S$ contained in U.*

This condition is satisfied, by all certainty equivalent functions of the expected-utility class, since the utility function u can be determined, up to an affine transformation, by behavior in arbitrarily small neighborhoods of the certainty ray. Similarly, in the case of Yaari's (1987) dual model, probability weights can be inferred from observation of arbitrarily small gambles, and this is sufficient to characterize behavior everywhere.

We can then obtain an immediate characterization for preferences that are twice differentiable

Proposition 6 : *Let a class E of symmetric certainty equivalents be characterized by preferences over small gambles and let $e \in E$ be twice differentiable. Then e is supermodular if and only if it is Schur-concave.*

Proof Consider the matrix of second derivatives evaluated at $\mu\mathbf{1}$, with typical elements

$$e_{st} = \frac{\partial^2 e(\mu\mathbf{1})}{\partial y_s \partial y_t}.$$

Symmetry requires that (i) for any s, t, $e_{ss} = e_{tt}$, and (ii) for any $q \neq r$, $s \neq t$, $e_{qr} = e_{st}$. That is, all diagonal entries of the matrix of second derivatives evaluated at $\mu\mathbf{1}$ are equal, as are all off-diagonal entries. Now consider the vector of first derivatives evaluated at $\mu\mathbf{1}$, with entries denoted e_s. Symmetry requires that all the e_s be equal, and the agreement property implies

$$\sum_s e_s = 1$$

so that $e_s = 1/S$ for all s, μ. Differentiating again, we obtain

$$e_{ss} + \sum_{t \neq s} e_{st} = 0.$$

Hence, for any $s, t, s \neq t$,

$$e_{st} = -\left(\frac{1}{S-1}\right) e_{ss}.$$

Hence, either $e_{ss} \leq 0$, $e_{st} \geq 0$, in which case the certainty-equivalent is supermodular and Schur-concave in a neighborhood of Δ, or $e_{ss} \geq 0$, $e_{st} \leq 0$ in which case the certainty-equivalent is submodular and Schur-convex in a neighborhood of Δ ■

Corollary 7 *Let preferences display constant relative risk aversion (radial homotheticity) on $\Re^S_+$ and let $e \in E$ be twice differentiable. Then e is supermodular if and only if it is Schur-concave.*

Proof Under constant relative risk aversion, preferences are completely characterized by their behavior in any neighborhood of $\mathbf{0}$ (in the relative topology inherited from $\Re^S$).■

5 Applications

In the state-contingent interpretation, the risky random variables appearing in economic analysis are simply vectors of state-contingent commodities, prices or income levels. Hence, comparative static results from standard consumer theory may be applied directly to the case of uncertainty. Topkis (1998, Section 2.8) gives a range of applications of supermodularity concepts to consumer theory. We now illustrate how similar arguments can be applied to problems of choice under uncertainty.

5.1 Asset demand

Consider a simple two-period asset demand problem with fixed initial wealth W, which may be consumed in period 0 or allocated to purchase securities yielding state-contingent income in period 1. The *ex ante* financial security payoffs are given by the $S \times J$ non-negative matrix $\mathbf{A}$, where $\mathbf{A}$ has full column rank. The prices of the financial securities are given by $\mathbf{v} \in \Re^J_+$. If the portfolio is denoted $\mathbf{h}$, period 1 state-contingent income is given by

$$\mathbf{y} = \mathbf{Ah}.$$

The individual chooses the portfolio, $\mathbf{h}$, to maximize

$$\max_{\mathbf{h}} \{e(\mathbf{Ah}) + (W - \mathbf{vh})\}.$$

If markets are complete, there exist unique state-claim prices $\mathbf{p} \in \Re^S$ satisfying

$$\mathbf{p}'\mathbf{A} = \mathbf{v}.$$

Thus, the portfolio selection problem can be rewritten as

$$\begin{aligned} & \max_{\mathbf{h}} \{e(\mathbf{Ah}) - \mathbf{p}'\mathbf{Ah}\} + W \\ = & \max_{\mathbf{y}} \{e(\mathbf{y}) - \mathbf{p}'\mathbf{y}\} + W. \end{aligned}$$

Applying Corollary 2.8.2 of Topkis (1998) now shows that, for a supermodular certainty equivalent, demand is decreasing in the state-claim prices. Hence, if e is supermodular, and thus, by Proposition 4, the individual is weakly risk-averse, the demand for each state-contingent income is decreasing in each state-claim price. Intuitively, it would not be surprising if increasing p_s should decrease the demand for y_s. When the certainty equivalent is supermodular, so that the individual is weakly risk-averse, state-contingent incomes are complementary, and thus as y_s decreases so do the other state-contingent demands, thus mitigating the dispersion of incomes across states of nature.

Because markets are complete, it follows trivially that

$$\mathbf{h} = \mathbf{A}^{-1}\mathbf{y},$$

so that comparative-static results for the optimal asset holdings can be derived from the monotone comparative static results implied by supermodularity of the certainty equivalent.

More generally, even if markets are not complete, there still exists a unique set of state-claim prices, $\hat{\mathbf{p}} \in \Re^S$, lying in the subspace generated by the columns of $\mathbf{A}$, which prices any asset lying in that subspace so that if $\mathbf{y} = \mathbf{Ah}$ for some $\mathbf{h}$, then

$$\hat{\mathbf{p}}'\mathbf{y} = \mathbf{v}'\mathbf{h}.$$

These prices are given by

$$\hat{\mathbf{p}}' = \mathbf{v}'(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'.$$

Thus, if investors face no background risk, then their portfolio choice decision in the absence of complete markets can be written as

$$= \max_{\mathbf{y}} \{e(\mathbf{y}) - \hat{\mathbf{p}}'\mathbf{y}\} + W.$$

Supermodularity of the certainty equivalent again leads to monotone comparative static results just as in the case of complete markets. Comparative static results for optimal asset holding are now available by combining the definition of $\hat{\mathbf{p}}$ with the recognition that the unique portfolio that yields asset position $\mathbf{y}$ is given by $(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{y}$.

5.2 Demand for insurance

A characterization of risk-aversion in terms of supermodularity and complementarity yields some immediate implications for behavior, independent of any hypothesis regarding functional form. Consider the proposition that demand for insurance against adverse events should increase with wealth. This proposition is consistent with ordinary economic intuition (namely, that insurance seems like a normal good) and with casual observation. On the other hand, the widely accepted hypothesis of decreasing absolute risk aversion generally implies the opposite.

We will focus on the first of these issues. Assuming constant absolute risk aversion to abstract from wealth effects, we can use supermodularity of the certainty equivalent to characterize the sense in which demand for insurance increases with wealth. The proof is analogous to Topkis's (1998, Example 2.8.1) treatment of the Le Chatelier principle.

Let $\mathbf{y} \in \Re^S$ denote the individual's state-contingent endowment of wealth and consider $\mathbf{m} \in \Re^S_+$ such that $\mathbf{m} > \mathbf{0}_s$ on some event $E \subseteq S$, and $\mathbf{m} = \mathbf{0}_s$ on E', the complement of E. Let $\mathbf{y}^{E'} \in \Re^S$ denote the endowment on E' so that

$$y_s^{E'} = \begin{cases} y_s & s \in E' \\ 0 & s \in E. \end{cases}$$

Define

$$v(\mathbf{m}; \mathbf{y}) = \sup \{ v \in \Re : e(\mathbf{y} + \mathbf{m} - v\mathbf{1}) \geq e(\mathbf{y}) \}.$$

Proposition 8 *Suppose e is supermodular and displays constant absolute risk aversion. Consider* $\mathbf{y}, \tilde{\mathbf{y}}$ *such that, for some* $\delta > 0$, $\mathbf{y}^E = \tilde{\mathbf{y}}^E$, $\mathbf{y}^{E'} \geq \tilde{\mathbf{y}}^{E'} + \delta\mathbf{1}$. *Then* $v(\mathbf{m}; \mathbf{y}) \geq v(\mathbf{m}; \tilde{\mathbf{y}})$.

Proof Apply supermodularity to $\tilde{\mathbf{y}}, \mathbf{y} + \mathbf{z}$. Observe that

$$\begin{aligned} (\mathbf{y}) \vee (\tilde{\mathbf{y}} + \mathbf{m}) &= \mathbf{y} + \mathbf{m} \\ (\mathbf{y}) \wedge (\tilde{\mathbf{y}} + \mathbf{m}) &= \tilde{\mathbf{y}}. \end{aligned}$$

Hence, supermodularity implies:

$$e(\mathbf{y} + \mathbf{m}) + e(\tilde{\mathbf{y}}) \geq e(\mathbf{y}) + e(\tilde{\mathbf{y}} + \mathbf{m})$$

or

$$e(\mathbf{y}+\mathbf{m})-e(\mathbf{y})\geq e(\tilde{\mathbf{y}}+\mathbf{m})-e(\tilde{\mathbf{y}}).$$

As shown by Quiggin and Chambers (1998), constant absolute risk aversion is equivalent to translation homotheticity of the certainty equivalent, that is, the requirement that, for all $\mathbf{y}$,δ

$$e(\mathbf{y}+\delta\mathbf{1})=e(\mathbf{y})+\delta.$$

Hence for any $v>0$, constant absolute risk aversion implies:

$$\begin{aligned}e(\mathbf{y}+\mathbf{m}-v\mathbf{1})-e(\mathbf{y}) &= e(\mathbf{y}+\mathbf{m})-e(\mathbf{y})-v\\ &\geq e(\tilde{\mathbf{y}}+\mathbf{m})-e(\tilde{\mathbf{y}})-v\\ &= e(\tilde{\mathbf{y}}+\mathbf{m}-v\mathbf{1})-e(\tilde{\mathbf{y}}).\blacksquare\end{aligned}$$

6 Concluding comments

The representation of random variables as state-contingent vectors provides a natural lattice structure to which the concepts of supermodularity theory may be applied. The certainty equivalent provides a natural cardinal representation of preferences under uncertainty. In this paper, we have shown that supermodularity of the certainty equivalent is a natural concept of risk aversion that is equivalent to the standard definition for the most commonly used models of choice under uncertainty, but does not depend on any specific functional form. We have also shown how these concepts may be related to the notion of arrangement increasingness. The applicability of supermodularity concepts to problems of comparative statics for risk-averse decision-makers has been illustrated by two simple examples.

7 References

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