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Linear-Risk-Tolerant, Invariant Risk Preferences

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Quiggin and Chambers (2004) have introduced the notion of invariant preferences. Briefly stated, preferences are invariant if the ranking of two state-contingent income vectors with equal means is not affected by either a change in base wealth or a change in the scaling of the vectors. These preferences generalize both the class of constant risk averse preferences (Safra and Segal, 1998; Quiggin and Chambers, 1998; Chambers and Quiggin, 2002) and mean–standard deviation preferences. In particular, Quiggin and Chambers (2004) show that preferences are invariant if and only if the certainty equivalent, e, assumes the general form:

$$e(\mathbf{y}) = \phi(\mu_{\hat{\pi}}(\mathbf{y}), \rho(\mathbf{y} - \mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})),$$

where $\mathbf{y}$ is a vector of state-contingent incomes, ϕ is a real-valued function increasing in its first argument and decreasing in its second, $\hat{\pi}$ is a given probability vector, $\mu_{\hat{\pi}}(\mathbf{y})$ is the mean of the state-contingent income vector evaluated with respect to $\hat{\pi}$, $\mathbf{1}$ is a vector of ones, and ρ is a nonnegative, lower semi-continuous, positively linearly homogeneous, and subadditive function. ρ, thus, generalizes the standard deviation, so that mean–standard deviation preferences are invariant.

Quiggin and Chambers (2004) show that the only invariant expected-utility functionals are those associated with a quadratic *ex post* utility function. This class of preferences has some very unattractive properties when regarded as preferences over wealth, but they also satisfy the conditions for two-fund portfolio separation and exhibit linear risk tolerance over a restricted domain. Invariant preferences always satisfy a form of two-fund portfolio separation in the presence of a riskless asset (Quiggin and Chambers 2004). This note identifies the class of preferences which simultaneously satisfy invariance, two-fund portfolio separation, and linear risk tolerance to determine if there exist meaningful classes of preferences, which inherit much of the quadratic family's theoretical and empirical tractability, but do not necessarily inherit its more unattractive properties when regarded as preferences over wealth.

Our analysis relies on the dual treatment of risk-averse preferences developed by Chambers and Quiggin (2002). In what follows, we first introduce some notation and basic concepts. Then we briefly discuss the translation and expected-value functions and then use these concepts to deduce necessary and sufficient conditions for individual preferences to be both invariant and linear-risk-tolerant. Finally, we consider implications for asset demand and asset pricing.

1 Notation and Basic Concepts

We consider preferences over random variables represented as mappings from a state space Ω to a convex outcome space $Y \subseteq \Re$. Ω is a finite set $\{1, ...S\}$, and the space of random variables is, thus, $Y^{\Omega} \subseteq \Re^{\Omega}$. The unit vector is denoted $\mathbf{1} = (1, 1, ...1)$, and $\mathcal{P} \subset \Re^S_+$ denotes the probability simplex. The vector $\hat{\pi} \in \mathcal{P}$ is taken to represent known (subjective or objective) probabilities over the state space.

Preferences over state-contingent incomes are given by the certainty equivalent $e(\mathbf{y})$, which is continuous, nondecreasing, and quasi-concave in $\mathbf{y}$. Quasi-concavity ensures that the least-as-good sets of the preference mapping

$$V(e) = \{\mathbf{y} : e(\mathbf{y}) \geq e\}$$

are convex, and that the individual is risk averse in the sense of Yaari (1969).

2 The Translation Function and the Expected-Value Function

The *translation function*, $B : \Re \times Y^S \rightarrow \Re$, is defined:

$$B(e, \mathbf{y}) = \max\{\beta \in \Re : \mathbf{y} - \beta\mathbf{1} \in V(e)\}$$

if $\mathbf{y} - \beta\mathbf{1} \in V(e)$ for some β, and $-\infty$ otherwise (Blackorby and Donaldson, 1980; Luenberger, 1992). [1] The properties of $B(e, \mathbf{y})$ are well known (Blackorby and Donaldson, 1980; Luenberger, 1992; Chambers, Chung, and Färe, 1996; Chambers and Quiggin, 2002).

[1] The translation function is a special case of the benefit function defined by Luenberger (1992).

Most importantly for our purposes, it is nonincreasing in e and nondecreasing and concave in $\mathbf{y}$.

We refer to the concave conjugate of the translation function, $B(e, \mathbf{y})$, as the *expected-value function* $E : \mathcal{P} \times \Re \to \Re$. It is defined by

$$E(\boldsymbol{\pi},e) = \inf_{\mathbf{y}} \{\boldsymbol{\pi}\mathbf{y} - B(e, \mathbf{y})\} \quad \boldsymbol{\pi} \in \mathcal{P}.$$

The expected-value function has an alternative interpretation as the expenditure function for $V(e)$ in the (normalized) state-claim prices $\boldsymbol{\pi}$ (Chambers, 2001; Chambers and Quiggin, 2002).

If $V(e)$ is nonempty, $B(e, \mathbf{y})$ is a continuous and nondecreasing proper concave function, and thus $E(\boldsymbol{\pi},e)$ is a closed, proper concave[2] function nondecreasing on $\mathcal{P}$ (Theorem 12.2, Rockafellar, 1970). It is also continuous and nondecreasing in e in the region where it is finite. And because $e(e\mathbf{1}) = e$, $E(\boldsymbol{\pi},e) \leq e$.

By basic results on conjugate duality (Theorem 12.2, Rockafellar, 1970), the translation function can be reconstructed from the expected-value function by applying the following conjugacy relationship

$$B(e, \mathbf{y}) = \inf_{\boldsymbol{\pi} \in \mathcal{P}} \{\boldsymbol{\pi}\mathbf{y} - E(\boldsymbol{\pi},e)\}.$$

3 Invariant Preferences

Quiggin and Chambers (2004) have shown that preferences are invariant if and only if there exists functions $\phi : \Re \times \Re_+ \to \Re$ and $\rho : \Re^S \to \Re_+$

$$e(\mathbf{y}) = \phi(\mu_{\hat{\pi}}(\mathbf{y}), \rho(\mathbf{y} - \mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})),$$

where $\mu_{\hat{\pi}}(\mathbf{y}) = \sum_s \hat{\pi}_s y_s$ for some $\hat{\boldsymbol{\pi}} \in \mathcal{P}$, ρ is nonnegative, lower semi-continuous, positively linearly homogeneous, and subadditive, and ϕ is increasing in its first argument and decreasing in its second and satisfies $\phi(c,0) = c$.

[2] A concave function, $g(x)$, is proper if there is at least one x such that $g(x) > -\infty$, and $g(x) < \infty$ for all x. A concave function is closed if and only if it is upper semi-continuous (Rockafellar, 1970, p. 52).

It is convenient to develop several preliminary definitions. Define the partial translation function for ϕ by

$$B^{\phi}(e,\mu,\rho)=\sup\left\{\beta:\phi(\mu-\beta,\rho)\geq e\right\}.$$

In words, $B^{\phi}(e,\mu,\rho)$ represents the number of units that μ must be translated to ensure that $\phi(\mu,\rho)$ is consistent with e. It follows immediately that for $\delta\in\Re$

$$B^{\phi}(e,\mu+\delta,\rho)=B^{\phi}(e,\mu,\rho)+\delta. \tag{1}$$

The monotonicity properties of ϕ ensure that $B^{\phi}(e,\mu,\rho')\leq B^{\phi}(e,\mu,\rho)$ for $\rho'\geq\rho$. Moreover, by the fact that $\phi(c,0)=c$

$$\begin{aligned}B^{\phi}(e,0,0) &= \sup\left\{\beta:\phi(-\beta,0)\geq e\right\}\\ &= \sup\left\{\beta:-\beta\geq e\right\}\\ &= -e.\end{aligned}$$

For the invariant preference class, there is a natural relationship between B and B^{ϕ}:

$$\begin{aligned}B(e,\mathbf{y}) &= \sup\left\{\beta:\phi\left(\mu_{\hat{\pi}}(\mathbf{y}-\beta\mathbf{1}),\rho(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})\right)\geq e\right\}\\ &= \sup\left\{\beta:\phi\left(\mu_{\hat{\pi}}(\mathbf{y})-\beta,\rho(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})\right)\geq e\right\}\\ &= B^{\phi}\left(e,\mu_{\hat{\pi}}(\mathbf{y}),\rho(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})\right).\end{aligned}$$

Because ρ is nonnegative, positively linearly homogeneous, and subadditive it is the gauge function for the convex set (Aliprantis and Border, Lemma 5.36)

$$C=\left\{\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}:\rho(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1})\leq 1\right\}.$$

Assuming that C is nonempty, define its (lower) support functional $R:\Re^{S}\rightarrow\Re$ by

$$R(\mathbf{p},C)=\inf_{\mathbf{y}}\left\{\mathbf{p}(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}):\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\in C\right\}.$$

With these definitions in hand we are able to state:

Lemma 1 *Risk-averse preferences are invariant if and only if the expected-value function can be written as*

$$E(\boldsymbol{\pi},e)=\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C)-B^{\phi}(e,0,\rho)\right\}.$$

Proof By (1)

$$\begin{aligned} B(e,\mathbf{y}) &= B^{\phi}\left(e,\mu_{\hat{\pi}}(\mathbf{y}),\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)\right) \\ &= \mu_{\hat{\pi}}(\mathbf{y})+B^{\phi}\left(e,0,\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)\right). \end{aligned}$$

Using this fact gives

$$\begin{aligned} E(\boldsymbol{\pi},e) &= \inf_{\mathbf{y}}\left\{\boldsymbol{\pi}\mathbf{y}-\boldsymbol{\mu}_{\hat{\pi}}(\mathbf{y})-B^{\phi}\left(e,0,\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)\right)\right\} \\ &= \inf_{\mathbf{y}}\left\{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\mathbf{y}-B^{\phi}\left(e,0,\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)\right)\right\} \\ &= \inf_{\mathbf{y},\rho>0}\left\{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\mathbf{y}-B^{\phi}(e,0,\rho):\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)=\rho\right\} \\ &= \inf_{\rho>0}\left\{\inf_{\mathbf{y}}\left\{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\mathbf{y}:\rho\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)=\rho\right\}-B^{\phi}(e,0,\rho)\right\} \\ &= \inf_{\rho>0}\left\{\inf_{\mathbf{y}}\left\{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\mathbf{y}:\rho\left(\frac{\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}}{\rho}\right)=1\right\}-B^{\phi}(e,0,\rho)\right\} \\ &= \inf_{\rho>0}\left\{\rho\inf_{\mathbf{y}}\left\{\frac{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\mathbf{y}}{\rho}:\rho\left(\frac{\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}}{\rho}\right)=1\right\}-B^{\phi}(e,0,\rho)\right\} \\ &= \inf_{\rho>0}\left\{\rho\inf_{\mathbf{y}}\left\{\frac{(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\left(\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}\right)}{\rho}:\rho\left(\frac{\mathbf{y}-\mu_{\hat{\pi}}(\mathbf{y})\mathbf{1}}{\rho}\right)=1\right\}-B^{\phi}(e,0,\rho)\right\} \\ &= \inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C)-B^{\phi}(e,0,\rho)\right\}. \end{aligned}$$

The fifth equality follows by the positive linear homogeneity of ρ, and the seventh follows by the fact that $(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}})\perp\mathbf{1}$ for all $\boldsymbol{\pi}\in\mathcal{P}$.■

It is worthwhile to remark on one particular aspect of this expected-value structure. Note first that:

$$\begin{aligned} E(\hat{\boldsymbol{\pi}},e) &= \inf_{\rho>0}\left\{\rho R(\hat{\boldsymbol{\pi}}-\hat{\boldsymbol{\pi}},C)-B^{\phi}(e,0,\rho)\right\} \\ &= -B^{\phi}(e,0,0) \\ &= e. \end{aligned}$$

Hence, in the terminology of Chambers and Quiggin (2002), an individual with invariant preferences is always risk-averse for the given probability vector $\hat{\boldsymbol{\pi}}$ that defines $\mu_{\hat{\pi}}(\mathbf{y})$. Visually, this implies that the least-as-good set for an individual with invariant preferences must have $\hat{\boldsymbol{\pi}}$ as a supporting hyperplane in the neighborhood of the ray $\mu\mathbf{1}$, $\mu\in\Re$.

Example 1 *Consider*

$$\phi\left(\mu_{\hat{\pi}}\left(\mathbf{y}\right),\rho\left(\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right)\right)=\mu_{\hat{\pi}}\left(\mathbf{y}\right)-g\left(\rho\left(\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right)\right)$$

with g increasing, $g(0)=0$, and ρ given by the L_p norm

$$\rho\left(\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right)=\left\|\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right\|_p$$

for $p\geq 1$. We adopt the the usual convention that $p=\infty$ corresponds to the supremum (Tchebycheff) norm. Thus, increases in p correspond to risk measures with a greater weight on extreme values. For example, $p=1$ yields the class of (sample) mean absolute deviation measures. Then

$$\begin{aligned}B\left(e,\mathbf{y}\right) &= B^{\phi}\left(e,\mu_{\hat{\pi}}\left(\mathbf{y}\right),\left\|\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right\|_p\right)\\ &= \mu_{\hat{\pi}}\left(\mathbf{y}\right)-g\left(\left\|\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right\|_p\right)-e,\end{aligned}$$

and

$$R\left(\mathbf{p},C\right)=\inf\left\{\mathbf{p}\left(\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right):\left\|\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right\|_p\leq\mathbf{1}\right\}.$$

Since it is characterized by a single parameter, and the special case $p=2$ corresponds to preferences characterized by the (sample) mean and standard deviation, this class of preferences is potentially suitable for empirical estimation. Finally,

$$\begin{aligned}E\left(\pi,e\right) &= \inf_{\mathbf{y}}\left\{\left(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}}\right)\mathbf{y}+g\left(\left\|\mathbf{y}-\mu_{\hat{\boldsymbol{\pi}}}\left(\mathbf{y}\right)\mathbf{1}\right\|_p\right)\right\}+e\\ &= \inf_{\rho>0}\left\{\rho R\left(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C\right)+g\left(\rho\right)\right\}+e.\end{aligned}$$

4 Linear Risk Tolerant and Invariant Preferences

Linear-risk-tolerant preferences are the class of preferences quasi-homothetic in state-contingent incomes. They, therefore, possess expected value functions of the form

$$E\left(\boldsymbol{\pi},e\right)=E^0\left(\boldsymbol{\pi}\right)+E^1\left(\boldsymbol{\pi}\right)e,$$

where E^0 and E^1 are expected value functions. We now have the main result of this paper:

Proposition 2 *Preferences are invariant and linear-risk-tolerant if and only if the expected-value function can be written as*

$$E(\boldsymbol{\pi},e) = (1-e)\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(0,0,\rho)\right\} + e\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(1,0,\rho)\right\}.$$

Proof For preferences to simultaneously exhibit linear risk tolerance and invariance, there must exist functions E^0, E^1, R, and B^{ϕ} satisfying

$$E^0(\boldsymbol{\pi}) + E^1(\boldsymbol{\pi})\,e = \inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(e,0,\rho)\right\}.$$

Set $e = 0$ to obtain

$$E^0(\boldsymbol{\pi}) = \inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(0,0,\rho)\right\}.$$

Setting $e = 1$ and using the last expression yields

$$E^1(\boldsymbol{\pi}) = \inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(1,0,\rho)\right\} - \inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(0,0,\rho)\right\}.$$

Therefore,

$$E(\boldsymbol{\pi},e) = (1-e)\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(0,0,\rho)\right\} + e\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(1,0,\rho)\right\}.$$

This establishes necessity. Sufficiency follows by conjugate duality.■

Preferences are invariant and linear-risk-tolerant if and only if the least-as-good set is the sum of rescaled (with the rescaling determined by e) versions of two invariant least-as-good sets. The first, which is dual to $\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(0,0,\rho)\right\}$ passes through the origin and has $\hat{\boldsymbol{\pi}}$ has a supporting hyperplane at that point. Hence, it is risk-averse for $\hat{\boldsymbol{\pi}}$. The second, which is dual to

$$\inf_{\rho>0}\left\{\rho R(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}},C) - B^{\phi}(1,0,\rho)\right\}$$

passes through the point $\mathbf{1}$ and has $\hat{\boldsymbol{\pi}}$ has a supporting hyperplane at that point. This observation offers a primal characterization of this class of preferences. The composition rules reported in Chambers, Chung, and Färe (1996) can be used along with this observation to deduce the associated translation function, and a representation of the certainty equivalent. However, because these preferences are consistent with quasi-homotheticity, there generally will not exist a closed form solution for the certainty equivalent.

5 Linear Risk Tolerance, Asset Demand and Asset Pricing

Quiggin and Chambers (2004) examine asset demand with invariant preferences, and show that for any given risk index ρ, and returns matrix $\tilde{\mathbf{Y}}$ including a riskless asset, there exists a portfolio $\boldsymbol{\alpha}^*$, $\sum_{j=2}^{N} \boldsymbol{\alpha}_i^* = 1$, such that, for all preferences of the form $e(\mathbf{y}) = \phi(E(\mathbf{y}), \rho(\mathbf{y}))$, and all wealth levels W, an interior solution to the portfolio problem exists and is of the form $(\alpha_1 \mathbf{e}^1 + (W - \alpha_1)\boldsymbol{\alpha}^*)$, where α_1 is the amount allocated to the riskless asset. In the presence of linear risk tolerance, we can impose the further restriction that

$$\alpha_1 = b_0 + b_1 W$$

for $W \geq b_0$.

In addition, we may derive implications for asset pricing. Suppose that all market participants have invariant and linear-risk-tolerant preferences, with the same risk index ρ. Then any market equilibrium must be associated with asset prices such that the return vector $\mathbf{y}^* = \tilde{\mathbf{Y}}\boldsymbol{\alpha}^*$ for the optimal portfolio is proportional to the state-contingent vector of aggregate returns for the market as a whole. Denote the return on the market portfolio by r_m and the return on the riskless asset by r_0. Hence the market risk premium is

$$r_m - r_0 = k\rho(\mathbf{y}^*)$$

where k is a parameter reflecting risk aversion.

As shown in Quiggin and Chambers (2003), invariance implies that the risk premium is of the form

$$\rho(\mathbf{y}) = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - \mathbf{E}(\mathbf{y})\mathbf{1}) : \mathbf{p} \in K^*\}$$

with K^* a closed convex set containing the origin.

Let

$$\mathbf{p}^* = \arg\max \{\mathbf{p}(\mathbf{y}^* - \mathbf{E}^*(\mathbf{y})\mathbf{1}) : \mathbf{p} \in K^*\},$$

be unique. Then, if the market portfolio yields $\mathbf{y}^*$, the change in risk associated with a small holding of any asset yielding return $\mathbf{y}$, with $\mathbf{E}(\mathbf{y}) = \mathbf{1}$, is given by the directional derivative

$$\rho'(\mathbf{y};\mathbf{y}^*) = \mathbf{p}^*\mathbf{y},$$

and the return required to hold $\mathbf{y}$ is

$$\begin{aligned} r &= r_0 + k\mathbf{p}^*\mathbf{y} \\ &= r_0 + \beta(r_m - r_0), \end{aligned}$$

where

$$\beta = \frac{\mathbf{p}^*\mathbf{y}}{\mathbf{p}^*\mathbf{y}^*}.$$

Hence, much of the standard mean–standard deviation analysis can be extended to general invariant preferences, without requiring the restrictive and unappealing assumption that preferences are neutral with respect to skewness and higher moments of the distribution of returns. However, because the Euclidean norm gives rise to an inner product, norms based on the standard deviation allow for a natural definition of notions such as covariance and correlation in terms of concepts of linear algebra. The corresponding analysis in the general case is nonlinear and thus may be difficult to cast in terms familiar from statistical analysis of the general linear model. However, it seems obvious that a generalized analysis of nonlinear models would permit a similar identification. One role of the specification of ρ could be to guide the direction that such nonlinear statistical analysis takes.

6 Conclusion

We have characterized in dual terms the family of preferences consistent with both invariant preferences and linear risk tolerance., and have given a parametric representation for a class of preferences with these properties.

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