Risk & Sustainable Management Group

Risk & Uncertainty Program Working Paper: 4/R04

Research supported by an Australian Research Council Federation Fellowship
http://www.arc.gov.au/grant_programs/discovery_federation.htm

Increasing Uncertainty: A Definition

Simon Grant
Rice University and Australian National University

and

John Quiggin
Australian Research Council Federation Fellow, University of Queensland

Schools of Economics and Political Science
University of Queensland
Brisbane, 4072
rsmg@uq.edu.au
http://www.uq.edu.au/economics/rsmg

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

15 May, 2004

Increasing Uncertainty: A Definition*

Abstract

We present a definition of increasing uncertainty, in which an elementary increase in the uncertainty of any act corresponds to the addition of an 'elementary bet' that increases consumption by a fixed amount in (relatively) 'good' states and decreases consumption by a fixed (and possibly different) amount in (relatively) 'bad' states. This definition naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty.

Keywords: uncertainty, ambiguity, risk, non-expected utility

JEL classifications: C72, D81

Simon Grant
Department of Economics
Rice University

John Quiggin
School of Economics
University of Queensland

Risk and Sustainable Management Group
Risk and Uncertainty Working Paper 6/R04

*We thank Anna Bogomolnaia and Klaus Nehring for their help and useful comments.

1 Introduction

Most formal analysis of economic decisions under uncertainty has relied on concepts of subjective probability. Significant advances in the discussion of preferences in the absence of well-defined subjective probabilities, and in understanding the relationship between preferences and subjective probabilities, have been made by Schmeidler (1989), Machina and Schmeidler (1992), Epstein (1999), Epstein and Zhang (2001) and Ghirardato and Marinacci (2002).

The analysis of economic decisions in the absence of well-defined subjective probabilities has often been referred to in terms of Knight's (1921) distinction between risk and uncertainty. However, Knight's discussion of the role of insurance companies and the Law of Large Numbers makes it clear that his conception of risk was confined to cases where objective probabilities can be defined in frequentist terms, and where risk can effectively be eliminated through pooling and spreading. All other cases, including those where individuals possess personal subjective probabilities, were effectively classed by Knight as involving uncertainty. The distinction now commonly drawn between 'risk' and 'uncertainty' could not be developed properly until the formulation of well-defined notions of subjective probability by de Finetti (1937) and Savage (1954).

The first writer to clearly identify cases where preferences were inconsistent with first-order stochastic dominance, relative to any possible probability distribution, was Ellsberg (1961) who distinguished between risk (subjective probabilities satisfying the Savage axioms) and ambiguity, leaving uncertainty as a comprehensive term. Therefore, consistent with the usage of Savage and Ellsberg, and with usage in the general economics literature, we will use the term uncertainty to encompass all decisions involving non-trivial state-contingent outcome vectors, whether or not the preferences and beliefs associated with these decisions can be characterized by well-defined subjective probabilities. Events for which subjective probabilities are (respectively, are not) well-defined will be referred to as 'unambiguous' (respectively, 'ambiguous') and problems involving acts measurable with respect to unambiguous events will be said to involve 'risk'. Our usage is consistent with Ghirardato and Marinacci (2002) and Epstein and Zhang (2001).

Epstein and Zhang (2001) provide a rigorous definition of ambiguous and unambiguous events, and lay the basis for an analysis of preferences under uncertainty, including both risk and ambiguity.[1] In light of this, the definition proposed by Epstein (1999) for a

[1] To the best of our knowledge, the only other definitions based solely on preferences are those provided by

comparative ambiguity aversion relation over preference relations can now be stated in a solely preference-based and functional-form free manner. However, questions of when one act is more uncertain or more ambiguous than another are not addressed in these analyses, except in the polar case where one act is ambiguous and the other is unambiguous. Ghirardato and Marinacci (2002) propose a *model-free* definition of *comparative uncertainty aversion*: one preference relation is more uncertainty averse than another, if whenever the latter relation expresses a weak preference for a constant act (that is, one that will yield the same outcome no matter what state of the world will obtain) over another act, then so must the former relation. They do not, however, consider the question of when one act is more uncertain than another except in the polar case where one of the acts yields a *certain* outcome.

By contrast, the concept of an increase in risk, and the economic consequences of increases in risk, have been analyzed extensively, beginning with the work of Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). These authors independently derived and characterized the second-order stochastic dominance condition (in terms of mean-preserving spreads), under which all risk-averse expected utility maximizers will prefer one probability distribution to another. Quiggin (1993) introduced an alternative notion of *monotone (mean-preserving) increase in risk*, defined in terms of co-monotonic random variables instead of mean-preserving spreads. Landsberger and Meilijson (1994) pointed out that this notion of increase in risk coincides with the Bickel and Lehmann (1976) notion of *dispersion of random variables with equal means.* Yaari (1969) argued that since any lottery is by definition a 'mean-preserving spread' of its mean, the weakest notion of risk aversion simply requires that the mean of a lottery for sure is weakly preferred to the lottery itself. Subsequent studies examined a wide range of generalizations of these stochastic dominance conditions, typically associated with more restrictive conditions on utility functions. Other papers that have extensively analyzed the concept of increasing risk in the context of generalized expected utility models include Chew, Karni and Safra (1987), Chateauneuf, Cohen and Meilijson (1997), Grant, Kajii and Polak (1992), Quiggin (1993) and Safra and Zilcha (1989).

Most concepts of increasing risk that have been considered in the literature are inher-

Sarin and Wakker (1998), Nehring (1999, 2001), Ghiradato and Marinacci (2002) and Ghiradato, Maccheroni and Marinacci (2004). In other papers such as Gilboa and Schmeidler (1994), Mukerji (1997) and Ryan (2002), the analysis focuses on a class of preference relations that admit a specific functional form. The criteria for what constitutes an ambiguous or unambiguous event is then defined in terms of a property or properties of the specific functional form representation that each of these preference relations admit.

ently dependent on the existence of well-defined subjective probabilities. This is obviously true of mean-preserving increases in risk, since the mean depends on probabilities. Even notions such as that of a compensated increase in risk (Diamond and Stiglitz, 1974), which do not depend on mean values, incorporate probabilities in their definitions. Yet the intuitive concept of an increase in the uncertainty of a prospect does not seem to depend crucially on probabilities. To take a simple example, doubling the stakes of a bet surely increases the uncertainty associated with that bet, regardless of whether the parties have well-defined and common subjective probabilities regarding the event that is the subject of the bet.

The main object of this paper is to examine concepts of increasing uncertainty, that are independent of any notion of subjective probabilities. A natural starting point is to consider whether existing concepts of 'elementary mean-preserving increases in risk', such as monotone spreads and Dalton transfers yield useful results when reference to probability distributions and means is dropped. We show that the monotone spread concept is robust to this generalization, but that concepts based on Dalton transfers, including the Rothschild-Stiglitz definition of increasing risk, have no content in the absence of well-defined probabilities. More precisely, the transitive closure of the analog of the Rothschild-Stiglitz definition turns out to be the trivial total ordering that includes every ordered pair of acts.

Any definition of increasing uncertainty naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty.

Proofs of the results, unless otherwise stated, appear in the appendix.

2 Preliminaries

Set-up and Notation. Denote by $\mathcal{S} = \{\ldots, s, \ldots\}$ a set of states and $\mathcal{E} = \{\ldots, A, B \ldots, E, \ldots\}$ the set of events which is a given σ-field on $\mathcal{S}$. We take the set of outcomes to be the set of non-negative real numbers, or 'consumption levels'. An act is a (measurable) real-valued and bounded function $f : \mathcal{S} \rightarrow \mathbb{R}_+$. Let $f(\mathcal{S}) = \{f(s) \mid s \in \mathcal{S}\}$ be the outcome set associated with the act f, that is, the range of f. Let $\mathcal{F} = \{\ldots, f, g, h, \ldots\}$ denote the set of acts on $\mathcal{S}$; and let $\mathcal{F}_0$ denote the set of simple acts on $\mathcal{S}$; that is, those with finite outcome sets. We will abuse notation and use x to denote both the outcome x in $\mathbb{R}_+$ and the constant act with $f(\mathcal{S}) = \{x\}$.

The following notation to describe an act will be convenient. For an event E in $\mathcal{E}$, and any two acts f and g in $\mathcal{F}$, let $f_E g$ be the act which gives, for each state s, the outcome $f(s)$ if s is in E and the outcome $g(s)$ if s is in the complement of E (denoted $\mathcal{S}\backslash E$).

In general, for any finite partition $\mathcal{P} := \{A^1, \ldots, A^n\}$ of $\mathcal{S}$ and any list of n acts $(h^1, \ldots, h^n)$, let $h^1_{A^1} h^2_{A^2} \ldots h^{n-1}_{A^{n-1}} h^n$ be the act that yields $h^i(s)$ if s is in A^i.

Let $\succsim$ be a binary relation over $\mathcal{F}$, representing the individual's preferences. Let $\succ$ and $\sim$ correspond to strict preference and indifference, respectively.

Given $\succsim$, for any act f in $\mathcal{F}$, we define the 'at least as good as f' set as the set $\succsim_f = \{g \in \mathcal{F} : g \succsim f\}$.

An event E is deemed *null* for the preference relation $\succsim$, if for all f and g in $\mathcal{F}$, $f_E g \sim g$.

We say a sequence of acts f_n *converges in the limit to f in the topology induced by the sup-norm*, written $f_n \to f$, if, $\lim_{n\to\infty} \sup_{s\in\mathcal{S}} |f_n(s) - f(s)| = 0$.

The only maintained assumptions we make on this preference relation is that it is a continuous preference ordering and satisfies a weak form of monotonicity.

Axiom 1 *The preference relation $\succsim$ is a continuous weak order: that is, it is transitive and complete and, for any of sequences of acts $\langle f_n \rangle$ and $\langle g_n \rangle$, such that $f_n \to f$ and $g_n \to g$, if $f_n \succsim g_n$ for all n, then $f \succsim g$.*

The following monotonicity axiom weaker than what is usually assumed is sufficient for our purposes.

Axiom 2 *The preference relation $\succsim$ is monotonic. That is, if for any pair of acts, f and g in $\mathcal{F}$, $f(s) \geq g(s) + \varepsilon$, with $\varepsilon > 0$, for all s in Ω, then $f \succ g$.*

We can prove that any preference relation $\succsim$ on $\mathcal{F}$ satisfying the axioms above may be characterized by a unique certainty equivalent of the form

$$m(f) = sup\{x \in \mathbb{R}_+ : f \succsim x\}.$$

2.1 An elementary increase in uncertainty

Under what circumstances may we view one act as being more uncertain than another? Given a probability measure exogeneously defined over the state space, it seems uncontroversial to denote any act as more risky than the constant act which yields the mean outcome of that act (evaluated according to that probability distribution) in every state. Other statistical partial orderings, such as second-order stochastic dominance or the Rothschild-Stiglitz definition of more risky, can also be invoked. However, in the absence of exogeneously given probabilities, it seems more natural to build up a 'more-uncertain-than' partial ordering over acts by considering the simplest operation that can be performed on an act that unequivocally increases the uncertainty associated with that act. The most elementary operation that we believe unequivocally increases the uncertainty associated with an act, is one that involves adding an 'elementary bet' to that act. The addition of an elementary bet increases consumption by a fixed amount in the (relatively) 'good' states and decreases consumption by a fixed (and possibly different) amount in the (relatively) 'bad' states. We refer to the addition of such a comonotonic elementary bet as an *elementary increase in uncertainty.*

Definition 1 *Fix a pair of acts $f, g \in \mathcal{F}$. The act g represents an* elementary increase in uncertainty *of the act f, denoted gUf if there exists a pair of positive numbers α and β, and an event $E^+ \in \mathcal{E} \backslash \{\mathcal{S}, \emptyset\}$ such that: (i) for all s in E^+, $g(s) - f(s) = \alpha$; (ii) for all s in $\mathcal{S} \backslash E^+$, $f(s) - g(s) = \beta$; and (iii)* $\sup\{f(s) : s \in \mathcal{S} \backslash E^+\} \leq \inf\{f(s) : s \in E^+\}$.

Correspondingly, we define a notion of comparative uncertainty aversion:

Definition 2 *Fix $\succsim$ and $\hat{\succsim}$. The preference relation $\succsim$ is at least as* uncertainty averse *at f as $\hat{\succsim}$ if for any gUf, $f \hat{\succsim} g$ implies $f \succsim g$. The preference relation $\succsim$ is* everywhere *at least as* uncertainty averse *as $\hat{\succsim}$ if for all f, $\succsim$ is as least as uncertainty averse at f as $\hat{\succsim}$.*

Notice that in the definition of an elementary increase in uncertainty there is no control made for "mean effects" as is usually the case for standard definitions in the context of exogeneously specified risk. This is because from the underlying primitives there is no way to define *independently of preferences* what is the mean of a elementary bet. Different individuals will find different elementary bets favorable or unfavorable depending on the context in which it takes place (that is, the 'base' act to which it is added) and their underlying preferences (which embodies their subjective assessment about the relative likelihood of different events obtaining.) In this context, if we see that whenever one individual finds unacceptable an elementary bet that has positive payoff in good states and negative payoffs in bad states then so does the other individual, then we refer to the latter as at least as unceraint averse as the former. So for example, if $\succsim$ is at least as *uncertainty averse* as $\hat{\succsim}$, then $100_A 50 \hat{\succsim} 1000_A 49$ requires $100_A 50 \succsim 1000_A 49$. That is, if $\hat{\succsim}$

with base contingent wealth $100_A 50$, finds the elementary bet (on the event A) of $900_A\,(-1)$ unacceptable, then so should $\succsim$. On the other hand, if $\hat{\succsim}$ finds the bet acceptable, then $\succsim$ may or may not find it acceptable since the definition is silent (at least directly) on the preference going one way or the other.

We can still define, however, a notion of *revealed* uncertainty neutrality. The underlying idea is that if an individual reveals a willingness to accept an elementary bet added to a particular base act, then if she is uncertainty neutral she should be willing to accept that same elementary bet added to *any* act.

Definition 3 *Fix $\succsim$. The preference relation $\succsim$ exhibits* uncertainty neutrality *if for any gUf, $g \succsim f$ implies $g' \succsim f'$ for any g', f' satisfying $g' - f' = g - f$.*

Correspondingly, we define a notion of absolute uncertainty aversion:

Definition 4 *Fix $\succsim$. The preference relation $\succsim$ is* uncertainty averse *if there exists a preference relation $\hat{\succsim}$ for which $\succsim$ is* everywhere *at least as* uncertainty averse *as $\hat{\succsim}$, and $\hat{\succsim}$ exhibits* uncertainty neutrality*.*

It is straightforward to see that uncertainty neutrality implies that, loosely speaking, the set of acceptable bets from a given act is the same no matter which act one starts with. If the state space is finite, then the only preference map over state-contingent wealth which satisfies this property is one in which the indifference sets are parallel hyperplanes and an analogous property holds for infinite state spaces. Intuitively, this means uncertainty neutrality is equivalent to saying the preference relation admits a subjective expected value representation. Formally, we have:

Proposition 1 *The preference relation $\succsim$ exhibits* uncertainty neutrality *if and only if it admits a* subjective expected value *representation. That is, there exists a probability measure π defined over $\mathcal{E}$, such that*

$$f \succsim g \Leftrightarrow \int_s f(s))\pi\,(ds) \geq \int_s g(s))\pi\,(ds)\,.$$

Hence a preference relation $\succsim$ is deemed uncertainty averse if it is more uncertainty averse than some subjective expected value maximizer.

In the next section we shall explore the implications of this definition both for sequences of bets and for particular classes of preferences.

3 Increases in uncertainty and uncertainty aversion

Our first observation about the definition of an elementary increase in uncertainty is that, no matter what assessment an individual attaches to any event (that may incorporate his or her belief and/or decision weight), an elementary increase in the uncertainty of a given act f always reduces consumption in the worst event and increases consumption in the best event. Furthermore, if gUf then g, f and the function $g-f$ are pairwise *co-monotonic* functions. That is, for every pair of states $s,t \in \mathcal{S}$,

$$(g(s) - g(t))(f(s) - f(t)) \geq 0$$

$$(g(s) - f(s) - g(t) + f(t))(f(s) - f(t)) \geq 0$$

$$(g(s) - g(t))(g(s) - f(s) - g(t) + f(t)) \geq 0.$$

As nothing in the above inequalities require the differences in question to be simple, we shall adopt these inequalities to define the *more uncertain* relation between *any* pair of acts.

Definition 5 *Fix a pair of acts $f,g \in \mathcal{F}$. The act g is* more uncertain *than the act f, denoted $g\overline{U}f$, if there exists a real-valued function h on $\mathcal{S}$, comonotonic with f such that* $\sup h > 0$, $\inf h < 0$ *and $g = f + h$.*

Our main result in this section is that the relation $\overline{U}$ is simply the transitive continuous closure of the relation U.

Proposition 2 *Fix a pair of acts $f,g \in \mathcal{F}$. If $g\overline{U}f$ then there exist sequences of simple acts, $\langle f_n \rangle$ and $\langle g_n \rangle$, such that $f_n \to f$ and $g_n \to g$, and for each n there exists a finite sequence of simple acts $\langle h_m^n \rangle_{m=1}^{M^n}$, such that $h_1^n = f_n$, $h_{M^n}^n = g_n$ and $h_{m+1}^n U h_m^n$, $m =$* $1, \ldots M^n - 1$.

The following is an immediate corollary of Proposition (2).

Corollary 3 *Fix $\succsim$ and $\widehat{\succsim}$. The preference relation $\succsim$ is* everywhere *at least as* uncertainty averse *as $\widehat{\succsim}$, if and only if,*

$$f \,\widehat{\succsim}\, g \textit{ implies } f \succsim g \textit{ for all } g\overline{U}f.$$

Also, we obtain

Corollary 4 *Any act f is more uncertain than its certainty equivalent $m(f)$.*

Corollary 5 *If $\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$, then for any f*

$$m(f) \leq \hat{m}(f).$$

From Corollary 5 it follows that if $\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$ then $\succsim$ is more uncertainty averse than $\widehat{\succsim}$ in the weaker sense of the following definition proposed by Ghirardato and Marinacci's (2002): the preference relation $\succsim$ is more (weakly) uncertainty averse than $\widehat{\succsim}$ if for any act f and any constant act x,

$$x \succsim f \Rightarrow x \mathrel{\widehat{\succsim}} f \text{ and } x \succ f \Rightarrow x \mathrel{\hat{\succ}} f$$

Ghirardato and Marinacci argue that their definition only relies upon the weakest prejudgement about what constitutes an unambiguous act, namely one that yields a given outcome for certain. Our definition encompasses this but goes further. Our rationale is that adding to an act a comonotonic simple bet should be considered by construction to have increased its uncertainty. Hence the natural definition for comparative uncertainty is the stronger one we propose in which a comonotonic simple bet being viewed unfavorably by an individual should entail that it is viewed unfavorably by any other individual who is more uncertainty averse.

Epstein (1999) proposed a definition of comparative *ambiguity aversion* that explicitly controled for 'risk aversion'. He did this by assuming that there was a rich set of exogeneously defined 'unambiguous events' $\mathcal{A} \subset \mathcal{E}$, that was closed under complementation and union. Any act that was measurable with respect to $\mathcal{A}$ was deemed an *unambiguous* act. The preference relation $\succsim$ is more ambiguity averse than $\widehat{\succsim}$ if for every unambiguous act h and every act f

$$h \succsim f \Rightarrow h \mathrel{\widehat{\succsim}} f \text{ and } h \succ f \Rightarrow h \mathrel{\hat{\succ}} f$$

Adopting Epstein and Zhang's (2001) purely behavioral definition for an unambiguous event, allows the outside analyst to compare two preference relations according to Epstein's definition without having to assume *a priori* which events are ambiguous or unambiguous.[2] We do not deny the usefulness of such an isolation of ambiguity aversion from

[2] Epstein and Zhang (2001) define an event T to be *unambiguous* if (a) for all disjoint subevents $A, B \subset \mathcal{S} \backslash T$, acts h, and outcomes x^*, x, z, z', $x^*_A x_B z_T h \succsim x_A x^*_B z_T h$ implies $x^*_A x_B z'_T h \succsim x_A x^*_B z'_T h$ and (b) the condition obtained if T is everywhere replaced by $\mathcal{S} \backslash T$ in (a) is also satisfied. Otherwise, T is *ambiguous.*

risk aversion where it can be achieved. But in circumstances where such a separation is not feasible, we believe there are useful insights and economic implications that can be drawn when comparisons according to the 'total' uncertainty aversion are made according to our definition of relative aversion to the addition of simple comonotonic bets.

3.1 Special cases

The definitions of comparative uncertainty, and of comparative uncertainty-aversion, presented above, do not depend on any specific features of the form of representation that a family of preference relations may or may not admit. It is of interest, however, to consider the case when preferences may be represented by some specific model, to characterize the relationship '$\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$' in terms of the parameters of that model, and, where appropriate, to compare that characterization to existing results on comparative risk aversion. We begin by demonstrating that the usual characterization of comparative risk aversion for subjective expected utility is consistent with our definition. More substantively, we analyze the cases of disappointment aversion (Gul 1991), and of Choquet Expected Utility preferences (Schmeidler 1989), incorporating such important special cases as rank-dependent expected utility under risk (Quiggin 1993) and the dual model of Yaari (1987).

3.1.1 Subjective Expected Utility

Let us consider the case when $\succsim$ and $\widehat{\succsim}$ satisfy the assumptions of Savage's theory of subjective expected utility (SEU). That is, assume both preference relations can be represented by certainty equivalent functionals $m, \widehat{m}$ of the form

$$m(f) = u^{-1}\left(\int_s u(f(s))\pi\,(ds)\right) \text{ and } \widehat{m}(f) = \widehat{u}^{-1}\left(\int_s \widehat{u}(f(s))\widehat{\pi}\,(ds)\right),$$

where π and $\widehat{\pi}$ are countably-additive and convex-ranged probability measures defined over $\mathcal{E}$, and u and $\widehat{u}$ are von Neumann-Morgenstern utility functions defined over $\mathcal{X}$.

The same set of necessary and sufficient conditions that are required for one preference relation to be at least as risk averse (in the sense of Rothschild and Stiglitz, 1970) as another are also necessary and sufficient for one to be at least as uncertainty averse as another.

Proposition 6 *Suppose $\succsim$ and $\widehat{\succsim}$ both admit SEU certainty equivalent representations $m()$ and $\widehat{m}(.)$, with associated probability measure and utility function pairs, (π, u) and*

$(\widehat{\pi}, \widehat{u})$, *respectively. Then,* $\succsim$ *is* everywhere *at least as* uncertainty averse *as* $\widehat{\succsim}$ *if and only if* $\pi(A) = \tilde{\pi}(A)$ *for all* $A \in \mathcal{E}$, *and* u *is a concave transform of* $\widehat{u}$

That is, under SEU, "more uncertainty averse" reduces to "common beliefs and more risk averse". An immediate corollary of Proposition (6) is that a necessary and sufficient condition for an SEU-maximizer to be averse to *monotone* mean-preserving spreads, that is, he is more uncertainty averse than the subjective expected value maximizer with probability distribution π, is that his utility function is concave. And without requiring any other restrictions, we also know that his preference relation would agree with the partial ordering of second-order stochastic dominance (or equivalently, he is averse to *all* mean-preserving spreads). These results are not surprising since it is well-known that under the expected utility model for decision making under risk (with exogenously specified probabilities) a decision maker is risk-averse in the weakest sense of always (weakly) preferring the mean of a lottery for sure to the lottery itself if and only if his utility index is concave. Such a coincidence of conditions necessary and sufficient for these three distinct notions of risk aversion (and their uncertainty analogs) does not hold in general for non-EU models of decision making under risk and non-SEU models of decision making under uncertainty. This point is illustrated by the following examples.

3.1.2 Disappointment Aversion

Disappointment aversion (Gul 1991) is the most widely used non-EU model displaying the "betweenness property" (see Chew 1983, Dekel 1986). In the context of the Savage framework a *subjective disappointment aversion (SDA)* functional representation, $V(f)$ may be implicitly defined by the equation

$$\sum_{x \in \mathbb{R}_+} \varphi\left(x, f^{-1}(x), V(f)\right) = 0.$$

where

$$\varphi(x, E, v) = \begin{cases} \mu(E)(1-b)(u(x) - v) & \text{if } u(x) \geq v \\ \mu(E)(u(x) - v) & \text{if } u(x) < v \end{cases},$$

where $\mu(.)$ is a probability measure defined on $\mathcal{E}$,

$u(.)$ is an increasing (utility) index and $b < 1$.

Notice that SEU is the special case in which $b = 0$. Gul implicitly assumes common subjective beliefs (i.e. $\pi = \widehat{\pi}$).

Gul (1991, Theorem 5, p676) shows that if $b \geq \widehat{b}$ and u is a concave transformation of $\widehat{u}$ then $\succsim$ is at least as risk averse (in the Yaari sense) as $\widehat{\succsim}$, that is, $m(f) \leq \hat{m}(f)$ for every f. It is straightforward to find counter-examples demonstrating that the converse does not hold.

As was the case for SEU, the same set of necessary and sufficient conditions that are required for one SDA preference relation to be at least as risk averse (in the sense of Rothschild and Stiglitz, 1970) as another are also necessary and sufficient for one SDA preference relation to be at least as uncertainty averse as another. To obtain a characterization of the necessary and sufficient conditions for comparative risk aversion in the sense of Rothschild and Stiglitz it is useful to define the following property

Definition 6 (Unboundedness) *For any act f, outcome c and non-null event E, there exists an outcome d sufficiently large that*

$$m\left(f_E d\right) \geq c$$

For the class of unbounded SDA preferences we can show the following equivalences hold.

Proposition 7 *Suppose $\succsim$ and $\widehat{\succsim}$ both satisfy Unboundedness and admit subjective disappointment aversion representations characterized by the two probability measure, utility function and disappointment parameter triples, (π, u, b) and $\left(\widehat{\pi}, \widehat{u}, \widehat{b}\right)$, respectively. Then, assuming $\succsim$ and $\widehat{\succsim}$ are distinct, the following three statements are equivalent:*

1. $\succsim$ *is* everywhere *at least as* uncertainty averse *as* $\widehat{\succsim}$;
2. $\succsim$ *is* at least as risk averse (in the Rothschild-Stiglitz sense) *as* $\widehat{\succsim}$;
3. $\pi(A) = \tilde{\pi}(A)$ *for all* $A \in \mathcal{E}$, *u is a concave transform of* $\widehat{u}$, $b \geq \widehat{b} = 0$

Notice in particular, that statement three of Proposition 7 implies that comparisons of uncertainty aversion and hence comparisons of risk aversion in the Rothschild-Stiglitz sense are only possible when one of the preference relations is SEU. This result does not depend on the existence of ambiguous events, and may therefore be seen as a limitation of SDA as a model of choice under risk.

3.1.3 Choquet Expected Utility

The other main direction for generalizing subjective expected utility has been the so-called "rank-dependent theories" of which Choquet Expected Utility (CEU) is the most widely applied. Associated with a CEU representation is an increasing utility index $u : X \to \mathbb{R}$ and a *capacity*, ν where a capacity is a function $\nu : \mathcal{E} \to [0,1]$ satisfying (i) for all A, B in $\mathcal{E}$: $\nu(A) \leq \nu(B)$, (ii) for any $\nu(\emptyset) = 0$; and (iii) $\nu(\mathcal{S}) = 1$. For such a CEU-maximizer $f \succsim g$ if and only if

$$\int_{-\infty}^{\infty} [\nu(\{s : u(f(s)) \geq w\}) - \nu(\{s : u(g(s)) \geq w\})]\, dw \geq 0 \tag{1}$$

In the context of decision making under risk, where preferences are defined over lotteries, Chateauneuf, Cohen and Meilijson (2004) provide a complete characterization for a Rank-Dependent Expected Utility (RDEU) maximizer to be averse to all monotone mean preserving increases in risk. To adapt their result to the subjectively uncertain act-framework fix a pair of CEU maximizers (u, ν) and $(\widehat{u}, \widehat{\nu})$. We shall assume throughout that both u and $\widehat{u}$ are continous and increasing. Furthermore, analogous to the non-atomic subjective beliefs for the case of subjective expected utility we shall also assume that both capacities ν and $\hat{\nu}$ are convex-valued, that is, for every $w \in (0,1)$, there exist events A and B, for which $\nu(A) = \hat{\nu}(B) = w$. Let

$$P_{(\nu,\widehat{\nu})} = \inf_{\{E \in \mathcal{E} : 0 < \widehat{\nu}(E) < 1\}} \frac{(1 - \nu(E))/\nu(E)}{(1 - \widehat{\nu}(E))/\widehat{\nu}(E)}$$

denote the *index of relative pessimism* of the capacity ν over the capacity $\widehat{\nu}$. We shall say the capacity ν is *relatively pessimistic* compared to the capacity $\widehat{\nu}$ if the latter is majorized by the former, that is, $\nu(E) \leq \widehat{\nu}(E)$, for all $E \in \mathcal{E}$, or equivalently, $P_{(\nu,\widehat{\nu})} \geq 1$. Let

$$G_{(u,\hat{u})} = \sup_{x^1 < x^2 \leq x^3 < x^4} \frac{[u(x^4) - u(x^3)] / [\hat{u}(x^4) - \hat{u}(x^3)]}{[u(x^2) - u(x^1)] / [\hat{u}(x^2) - \hat{u}(x^1)]}$$

denote the *index of relative greediness* of the utility function u over the utility function $\hat{u}$. This index satisfies $G_{(u,\hat{u})} \geq 1$ and will equal 1 if u is a monotonic concave transformation of $\hat{u}$. When u and $\hat{u}$ are differentiable, it is the supremal value of $u'(x)\,\hat{u}(y) / [u'(y)\,\hat{u}(x)]$ taken over $x \geq y$.

Relying heavily on Chateauneuf, Cohen and Meilijson's (2004, Theorem 1) characterization result we obtain the following characterization.[3]

[3] A model that is closely related to CEU is cumulative prospect theory (CPT). It is more general as

Proposition 8 *A CEU maximizer, (u, ν), is at least as uncertainty averse as another CEU maximizer, $(\widehat{u}, \widehat{\nu})$ if and only if her index of relative pessimism exceeds her index of relative greediness: $P_{(\nu,\widehat{\nu})} \geq G_{(u,\widehat{u})}$.*

One implication of this result is that a CEU maximizer with a non-concave utility index can be more uncertainty averse than a Yaari-CEU maximizer (that is, a CEU maximizer with a linear utility function) or even a CEU maximizer with a strictly concave utility index provided the degree of 'pessimism' embodied in his capacity, as measured by the ratio $(1 - \nu(E)) / \nu(E)$, is sufficiently strong enough to outweigh any region of non-diminishing marginal utility. Again, this accords with similar results derived in the context of decision making under risk for RDEU maximizers. And we obtain the following characterization for an uncertainty averse CEU maximizer.

Corollary 9 *A CEU maximizer, (u, ν), is uncertainty averse if and only if there exists a probability measure π, for which*

$$\inf_{\{E \in \mathcal{E}: 0 < \widehat{\nu}(E) < 1\}} \frac{(1 - \nu(E)) / \nu(E)}{(1 - \pi(E)) / \pi(E)} \geq \sup_{x^1 < x^2 \leq x^3 < x^4} \frac{[u(x^4) - u(x^3)] / [x^4 - x^3]}{[u(x^2) - u(x^1)] / [x^2 - x^1]}$$

3.1.4 Biseparable Preferences

All the families of preferences considered so far are themselves special cases of the class of preferences Ghirardato and Marinacci (2001) dub *biseparable.* Essentially this is the class of preferences relations whose restriction to the set of acts that are measurable with respect to a two-element partition of the state space admits a CEU representation. That is, $\succsim$ is biseparable if there exists an increasing utility index $v : X \to \mathbb{R}$ and a *capacity*, ρ, such that for any $(A, B) \in \mathcal{E} \times \mathcal{E}$ and any $(w, x, y, z) \in \mathbb{R}^4$, such that $w \geq x$ and $y \geq z$,

$$w_A x \succsim y_B z \Leftrightarrow \rho(A) v(w) + [1 - \rho(A)] v(x) \geq \rho(B) v(y) + [1 - \rho(B)] v(z).$$

As Ghirardato and Marinacci observe, this is the most general model achieving a separation of 'cardinal' state-independent utility and a unique representation of beliefs. Not only does it encompass, SEU, SDA and CEU but Gilboa and Schmeidler's (1989) 'maxmin expected utility' (or 'multiple priors') is included in this class. In the multiple prior model,

it allows for reference-dependence. Utility is defined on deviations from a 'status-quo' outcome and the capacity exhibits 'sign-dependence', depending on whether the best outcome on an event is better or worse than the status quo. But modulo the necessary adjustments for reference-dependence, analogous results to the ones we derive for the CEU model hold for CPT.

the preference relation admits a representation consisting of an increasing utility function u and a convex set of 'prior' probability distributions Π, so that

$$f \succeq g \Leftrightarrow \min_{\pi \in \Pi} \int_s u(f(s))\pi\,(ds) \geq \min_{\pi \in \Pi} \int_s u(g(s))\pi\,(ds)\,.$$

Clearly, setting $v\,(x) := u\,(x)$ and $\rho\,(E) = \min_{\pi \in \Pi} \pi\,(E)$, yields the biseparable representation.

Now since the proof of the necessity part of Proposition (8) in the Appendix only utilizes acts that are measurable with respect to some two-element partition of the state space, we have the as an immediate corollary of Proposition (8) the following.

Corollary 10 *A necessary conditon for one preference relation $\succsim$ that admits a biseparable representation (v, ρ), where ρ is convex-valued and u is continuous and increasing, to be at least as uncertainty averse as another preference relation $\hat{\succsim}$ that admits a biseparable representation $(\hat{v}, \hat{\rho})$, where $\hat{\rho}$ is convex-valued and $\hat{v}$ is continous and increasing, is that $P_{(\rho,\hat{\rho})} \geq G_{(v,\hat{v})}$.*

Whether this is also sufficient for the various sub-classes of biseparable preferences is an open question. In particular, we do not know whether it is sufficient for the multiple prior model. It is sufficient for SEU, the condition entails that $P_{(\pi,\hat{\pi})} = 1$, and moreover, that $\pi\,(E) = \hat{\pi}\,(E)$ for every event E, hence $G_{(u,\hat{u})} = 1$, which recall implies that u is a concave transformation of $\hat{u}$. It is not sufficient for SDA, as we saw sufficiency required that at least one of the two preference relations was SEU.

3.1.5 Subjectively Ambiguous and Unambiguous Events

A further particularly interesting application of Proposition (8), is in the context of Epstein and Zhang's (2001) model of a CEU maximizer, (u, ν), for whom, just from the behavioral implications of the preference relation, an outside analyst is able to classify each event as being either 'ambiguous' or 'unambiguous' for that preference relation. Let $\mathcal{E}_\nu^{UA} \subset \mathcal{E}$, denote the set of unambiguous events for (u, ν). The set of axioms that they impose on the preference relation guarantees that the set of 'unambiguous' events is rich enough so that 'beliefs' over these events can be represented by a countably additive, convex-valued probability measure $\pi : \mathcal{E}^{UA} \to [0, 1]$. Moreover, for each $A \in \mathcal{E}^{UA}$, $\nu\,(A) = \phi\,(\pi\,(A))$, for some strictly increasing and onto map $\phi : [0, 1] \to [0, 1]$. Hence for any (measurable) finite partition, $(A^1, \ldots, A^n)$, and for all acts of the form $f = x^1_{A^1} x^2_{A^2} \ldots x^{n-1}_{A^{n-1}} x^n$ for which $x^1 \geq$

$\ldots \geq x^n$, the certainty equivalent function, $m(f)$ for the CEU maximizer, (u, ν), is defined by:

$$m(f) = u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\nu\left(\cup_{j=1}^{i}A^j\right) - u\left(x^i\right)\nu\left(\cup_{j=1}^{i-1}A^j\right)\right)\right).$$

Furthermore, if for each $i = 1, \ldots, n$, $A^i \in \mathcal{E}^{UA}$, then f is an unambiguous act and

$$m(f) = u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\phi\left(\pi\left(\cup_{j=1}^{i}A^j\right)\right) - u\left(x^i\right)\phi\left(\pi\left(\cup_{j=1}^{i-1}A^j\right)\right)\right)\right).$$

If we take another such CEU maximizer $(u, \widehat{\nu})$, for whom $\mathcal{E}_{\widehat{\nu}}^{UA} = \mathcal{E}$ (that is, every event is unambiguous for this individual) and $\widehat{\nu}(A) = \nu(A)$ for every $A \in \mathcal{E}_{\nu}^{UA}$, then by construction the two CEU maximizers, (u, ν) and $(u, \widehat{\nu})$, agree over any pair of acts that are measurable with respect to $\mathcal{E}_{\nu}^{UA}$. Furthermore, since every event is unambiguous for $(u, \widehat{\nu})$, this CEU-maximizer is *probabilistically sophisticated* in the sense of Machina and Schmeidler (1992), and so corresponds to Epstein's (1999) notion of an *ambiguity neutral* preference relation. Thus there exists a countably additive, convex-valued probability measure, $\widehat{\pi}$ that extends π to $\mathcal{E}$. That is, for any $E \in \mathcal{E}$, $\widehat{\nu}(E) = \phi(\widehat{\pi}(E))$, and for any (measurable) finite partition, $(A^1, \ldots, A^n)$, and for all acts of the form $f = x_{A^1}^1 x_{A^2}^2 \ldots x_{A^{n-1}}^{n-1} x^n$ for which $x^1 \geq \ldots \geq x^n$, the certainty equivalent function, $\widehat{m}(f)$ for the CEU maximizer, $(u, \widehat{\nu})$, is defined by:

$$\begin{aligned}
\widehat{m}(f) &= u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\widehat{\nu}\left(\cup_{j=1}^{i}A^j\right) - u\left(x^i\right)\widehat{\nu}\left(\cup_{j=1}^{i}A^{j-1}\right)\right)\right) \\
&= u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\phi\left(\widehat{\pi}\left(\cup_{j=1}^{i}A^j\right)\right) - u\left(x^i\right)\phi\left(\widehat{\pi}\left(\cup_{j=1}^{i}A^{j-1}\right)\right)\right)\right).
\end{aligned}$$

According to Epstein's (1999) definition, (u, ν) is *ambiguity averse* if for any pair of acts f and h, such that h is measurable with respect to $\mathcal{E}_{\nu}^{UA}$,

$$\widehat{m}(h) \geq \widehat{m}(f) \text{ implies } m(h) \geq m(f).$$

Epstein and Zhang (2001) show that (u, ν) is *ambiguity averse* if and only if

$$\widehat{\pi}(E) \geq \phi^{-1}(\nu(E)) \text{ for all } E \in \mathcal{E}.$$

The following corollary to Proposition (8) establishes the connection of our definition of more uncertainty averse to Epstein's (1999) definition of ambiguity averse.

Corollary 11 *Let* $\succsim$ *and* $\widehat{\succsim}$ *be preference relations corresponding to the CEU-maximizers* (u,ν) *and* $(u,\widehat{\nu})$ *defined above. Then* (u,ν) *is* ambiguity averse *in the sense of Epstein (1999), if and only if* $\succsim$ *is more uncertainty averse than* $\widehat{\succsim}$.

From this corollary we can conclude that a CEU-maximizer is ambiguity averse in the sense of Epstein (1999) if and only if there is a probabilistically sophisticated CEU-maximizer, such that: (a) the two preference relations agree over the set of unambiguous acts; and (b) the former is more uncertainty averse than the latter.

4 Alternative notions of elementary increases in risk and uncertainty

The definition of an elementary increase in uncertainty presented above is the simplest possible. As we have shown, its transitive closure is a monotone spread. This result also holds under risk (see Quiggin, 1993). There are, however, a wide range of alternative notions of increasing risk. Chateauneuf, Cohen and Meilijson (2002) give a summary, and their discussion suggests a systematic procedure for generating various classes of increases in risk as the transitive closure of appropriate notions of an elementary increase in risk. In addition to monotone spreads, Chateauneuf, Cohen and Meilijson consider the Rothschild-Stiglitz mean-preserving riskier ordering and two intermediate orderings, referred to as left-monotone and right-monotone increases in risk.

In the case of risk, these intermediate orderings can be generated as transitive closures of elementary increases in risk based on the following notion of three-event *ordered* partitions of the state space.

Definition 7 *Fix an act* f*. The* N*–event partition* $\left\{E^1,\ldots,E^N\right\}$ *of* $\mathcal{S}$ *is* ordered *with respect to* f *if for all* $n=1,\ldots,N-1$

$$\sup\left\{f\left(s\right):s\in E^n\right\}\leq\inf\left\{f\left(s\right):s\in E^{n-1}\right\}.$$

Definition 8 *An act* g *is a elementary left-increase in uncertainty on* f *if there exist numbers* α *and* β *and a* 3*–event partition* $\{E^1,E^2,E^3\}$ *that is ordered with respect to* f *and* g*, such that*

$$g(s)=\begin{cases} f\left(s\right)-\beta & s\in E^1\\ f\left(s\right)+\alpha & s\in E^2\\ f(s) & s\in E^3\end{cases}$$

Definition 9 *An act g is a elementary right-increase in uncertainty on f if there exist numbers α and β and a 3–event partition $\{E^1, E^2, E^3\}$ that is ordered with respect to f and g, such that*

$$g(s) = \begin{cases} f(s) & s \in E^1 \\ f(s) - \beta & s \in E^2 \\ f(s) + \alpha & s \in E^3 \end{cases}$$

Definition 10 *An act g is a monotone increase in uncertainty on f if there exist numbers α and β and a 3–event partition $\{E^1, E^2, E^3\}$ that is ordered with respect to f such that*

$$g(s) = \begin{cases} f(s) - \beta & s \in E^1 \\ f(s) & s \in E^2 \\ f(s) + \alpha & s \in E^3 \end{cases}$$

As the terminology suggests, if g is a monotone increase in uncertainty on f, then g is more uncertain than f in the sense of Definition 5. Indeed, for any $\gamma < \min(\alpha, \beta)$, we can generate a monotone increase in uncertainty from the sequence of elementary increases in uncertainty, $h^1 = f$, $h^2(s) = f(s) - \beta + \gamma$, $s \in E^1$, $h^2(s) = f(s) + \gamma$, $s \in E^2 \cup E^3$, $h^3 = g$, that is $h^3(s) = h^2(s) - \gamma, s \in E^1 \cup E^2, h^2(s) = h^3(s) + \alpha - \gamma, s \in E^3$.

Using the results of Chateauneuf, Cohen and Meilijson (Lemma 2, p11), it is straightforward to show that, if we assume known probabilities, and add the requirement that g has the same expected value as f, the transitive closure of the class of elementary left-(respectively, right-)increases in uncertainty on f is the class of acts that are ranked left-(respectively, right-)monotone more riskier than f. Also observe that an elementary increase in uncertainty satisfies each of the definitions 8 , 9 and 10 with the 'unchanged' event being empty.

Now consider potential notions of elementary increases in uncertainty of an act with respect to a four-event ordered partition. The only elementary operation that is (i) measurable with respect to a four-event partition, (ii) does not include an increase on E^i and a reduction in E^j, for all $j > i$, and (iii) cannot be generated by a finite sequence of any of the elementary increases considered above, is the following.

Definition 11 *An act g is an elementary conditional-increase in uncertainty on f if there exist numbers α and β and a 4–event partition $\{E^1, E^2, E^3, E^4\}$ that is ordered with respect to f and g, such that*

$$g(s) = \begin{cases} f(s) & s \in E^1 \\ f(s) - \beta & s \in E^2 \\ f(s) + \alpha & s \in E^3 \\ f(s) & s \in E^4 \end{cases}$$

Observe that elementary, left-elementary and right-elementary increases in uncertainty are all special cases of elementary conditional-increase in uncertainty. Moreover, if we assume known probabilities, and add the requirement $E[g] = E[f]$, the transitive closure of the relation 'g is derived from f' by an elementary conditional-increase in uncertainty' is the relation 'g is mean-preserving riskier than f' in the sense of Rothschild-Stiglitz.

Our main result in this section is that no such relationship applies under uncertainty. In fact, if we suppose that the state-space is sufficiently rich, in the sense that there are no "atoms" in the state space, then it follows that the transitive closure is the trivial total ordering, which includes *every* ordered pair of acts. That is, suppose we require in addition to the other maintained assumptions that the preference relation satisfies Savage's postulate *small-event continuity.*

Definition 12 *The relation* $\succsim$ *exhibits* small event continuity *if for any pair of acts* $f \succ g$*, and any outcome* x*, there exists a finite partition of the state space* $\{E^1, \dots, E^N\}$ *such that* $x_{E^n} f \succ g$ *and* $f \succ x_{E^n} g$ *for every* $n = 1, \dots, N$*.*

Proposition 12 *Suppose* $\succsim$ *exhibits* small event continuity. *The transitive closure of the relation 'g is derived from f by an elementary conditional-increase in uncertainty' is the full relation* $\mathbf{R}$ *in which, for all pairs of act* f *and* g*,* $g\mathbf{R}f$ *and* $f\mathbf{R}g$*.*

Thus, there is no non-trivial analog under uncertainty for the Rothschild-Stiglitz notion of an increase in risk.

5 Conclusion

Most economic analysis of choice under uncertainty, and particularly of increases in uncertainty, has been based on the assumption that decision-makers have well-defined subjective probabilities. On the other hand, the fundamental result of the literature, the proof of existence of equilibrium in state-contingent markets derived by Arrow and Debreu (1954), does not require decision-makers to possess subjective probabilities or to satisfy the postulates of any model specific to problems involving uncertainty.

In this paper, we have shown that some, but not all, of the concepts that have been used in the case of known probabilities can be extended to the more general and realistic case of unknown probabilities. Broadly speaking, concepts that are most naturally expressed in state-continent terms, such as statewise dominance and monotone spreads, are robust.

Concepts that are most naturally expressed in terms probabilities or cumulative probability distributions, such as notions of stochastic dominance, are unlikely to be robust.

Appendix

Proof of Proposition 2.

We first establish the following lemmas.

Lemma 13 *If for any pair of simple acts g and f, any pair of positive real numbers, α and β, and any three element partition of $\mathcal{S}$, (E_{-1}, E_0, E_1) , we have*

$$g(s) - f(s) = \begin{cases} \alpha & \text{if } s \in E_1 \\ 0 & \text{if } s \in E_0 \\ -\beta & \text{if } s \in E_-1 \end{cases}$$

then there exists a simple act h for which gUh and hUf.

Proof: If $\alpha > \beta/2$ then define

$$h(s) = \begin{cases} f(s) + \alpha - \beta/2 & \text{if } s \in E_1 \\ f(s) - \beta/2 & \text{if } s \in E_0 \\ f(s) - \beta/2 & \text{if } s \in E_{-1}. \end{cases}$$

Notice that

$$g(s) - h(s) = \begin{cases} \beta/2 & \text{if } s \in E_1 \cup E_0 \\ -\beta/2 & \text{if } s \in E_{-1} \end{cases} \quad \text{and } h(s) - f(s) = \begin{cases} \alpha - \beta/2 & \text{if } s \in E_1 \\ -\beta/2 & \text{if } s \in E_0 \cup E_{-1} \end{cases}$$

as required. If $\alpha \leq \beta/2$ then define

$$h(s) = \begin{cases} f(s) + \alpha/2 & \text{if } s \in E_1 \\ f(s) + \alpha/2 & \text{if } s \in E_0 \\ f(s) - \beta + \alpha/2 & \text{if } s \in E_{-1}. \end{cases}$$

Now we have

$$g(s) - h(s) = \begin{cases} \alpha/2 & \text{if } s \in E_1 \\ -\alpha/2 & \text{if } s \in E_0 \cup E_{-1} \end{cases} \quad \text{and } h(s) - f(s) = \begin{cases} \alpha/2 & \text{if } s \in E_1 \cup E_0 \\ -\beta + \alpha/2 & \text{if } s \in E_{-1}. \end{cases}$$

■

Lemma 14 *If for any pair of simple acts f and g, $g\overline{U}f$ then there exists a finite sequence of simple acts $\langle h_m \rangle_{m=1}^{M}$ such that $h_1 = f$, $h_M = g$ and $h_{m+1}Uh_m$, $m = 1, \ldots M-1$.*

Proof: From the definition of $g\overline{U}f$ it follows that $g - f$ is pairwise co-monotonic with both g and f. Let $[E_{-J}, \ldots, E_1, E_0, E_1, \ldots, E_I]$ be the coarsest partition of $\mathcal{S}$ for which $g - f$ is measurable and with the labelling monotonically ordered, that is for any $i > j$, and any $s \in E_i$ and $s' \in E_j$, $g(s) - f(s) > g(s') - f(s')$. Moreover, assume that for any $i < 0$, and any $s \in E_i$, $g(s) - f(s) < 0$;. for any $i > 0$ and any $s \in E_i$, $g(s) - f(s) > 0$; and for any $s \in E_0$, $g(s) = f(s)$ E_0 may be empty, but since $\inf_{s \in \mathcal{S}} g(s) < \inf_{s \in \mathcal{S}} f(s)$ and $\sup_{s \in \mathcal{S}} g(s) > \sup_{s \in \mathcal{S}} f(s)$ it follows that $I \geq 1$ and $J \geq 1$. For each $i = -J, \ldots, 0, \ldots, I$, and some $s_i \in E_i$, set $d_i := g(s_i) - f(s_i)$. By construction, we have

$$d_{-J} < d_{-J+1} < \ldots < d_{-1} < d_0 = 0 < d_1 < \ldots < d_I.$$

Let $h_1 := f$. Define

$$h_3(s) = \begin{cases} f(s) + d_1 & \text{if } s \in E_1 \cup E_2 \cup \ldots \cup E_I \\ f(s) & \text{if } s \in E_0 \\ f(s) + d_{-1} & \text{if } s \in E_{-1} \cup E_{-2} \cup \ldots \cup E_{-J}. \end{cases}$$

For $i = 2, \ldots, \min\{I, J\} - 1$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_i \cup \ldots \cup E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\ h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J}. \end{cases}$$

$I \geq J$, then for $i = J, \ldots, I$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_i \cup \ldots \cup E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-J+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\ h_{2i-1}(s) + (d_{-J} - d_{-J+1}) / (I - J + 1) & \text{if } s \in E_{-J}. \end{cases}$$

Notice, in this case $h_{2I+1} = g$.

If, however, $I < J$, then for $i = I, \ldots, J$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + (d_I - d_{I-1}) / (J - I + 1) & \text{if } s \in E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{I-1} \\ h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J} \end{cases}$$

and now $h_{2J+1} = g$.

For each $i = 1, \ldots, \max\{I, J\}$, it follows from Lemma (13) that there exists a simple act h_{2i} for which $h_{2i+1} U h_{2i}$ and $h_{2i} U h_{2i-1}$. Hence we have

$$g = h_{2\max\{I,J\}+1} U h_{2\max\{I,J\}} U \ldots U h_1 = f$$

as required.

We are now in a position to prove the proposition. Let f_n and h_n be the usual uniform simple approximations from below of f and h; for n large enough $\sup h_n > 0$ and $\inf h_n < 0$. Moreover, as noted in Chateauneuf, Cohen and Meilijson (1997), f and h comonotonic implies that f_n and h_n are comonotonic, and therefore result follows from Lemma (14). ■

Proof of Proposition 6.

Sufficiency is obvious. For necessity of the equality of π and $\widehat{\pi}$, consider choices in a neighborhood of a constant act x. For any real-valued function $d : \mathcal{S} \to \mathbb{R}$ and sufficiently small $\varepsilon > 0$, the certainty equivalent of the act $x + \varepsilon d$ (in the neighborhood of x) under m is approximately

$$x + \varepsilon \int_s d(s) \pi(ds)$$

and, similarly for $\widehat{m}$, the certainty equivalent is

$$x + \varepsilon \int_s d(s) \widehat{\pi}(ds).$$

If $\pi(E) > \tilde{\pi}(E)$ for some $E \subset \mathcal{S}$ then

$$\frac{\pi(E)}{1-\pi(E)} > \frac{\widehat{\pi}(E)}{1-\widehat{\pi}(E)} - \frac{\delta}{(1-\pi(E))(1-\widehat{\pi}(E))}$$

for some $\delta > 0$. Thus if we take

$$d(s) = \begin{cases} 1-\widehat{\pi}(E)-\delta & \text{if } s \in E \\ \\ -\widehat{\pi}(E)-\delta & \text{if } s \notin E, \end{cases}$$

then for any $\varepsilon > 0$ we have

$$\varepsilon \int_s d(s)\pi(ds) > 0 > \varepsilon \int_s d(s)\widehat{\pi}(ds)$$

and $x + \varepsilon d\, \overline{U}\, x$. So for sufficiently small $\varepsilon > 0$, it follows from continuity and monotonicity of $\succsim$ and $\widehat{\succsim}$, that $x \,\widehat{\succ}\, (x+\varepsilon d)$ but $(x+\varepsilon d) \succ x$.

To demonstrate the necessity of u being a concave transformation of $\widehat{u}$, suppose the contrary, that is, u is not a concave transform of $\tilde{u}$. Then there must exist utility levels v_1, v_2 and v_3 in the *range* of $\widehat{u}$, and λ in $(0,1)$, such that

$$\lambda v_1 + (1-\lambda) v_3 = v_2$$
$$\lambda u \circ \widehat{u}^{-1}(v_1) + (1-\lambda) u \circ \widehat{u}^{-1}(v_3) > u \circ \widehat{u}^{-1}(v_2).$$

Since π is non-atomic, there exists an event $E \subset \mathcal{S}$ for which $\pi(E) = \lambda$. So consider the act $f := \widehat{u}^{-1}(v_1)_E\, \widehat{u}^{-1}(v_3)$ and the constant act $x := \widehat{u}^{-1}(v_2)$. By construction we have f $\overline{U}$ x, $x \,\widehat{\succsim}\, f$ and $f \succ x$ ■

Proof of Proposition 7.

Proof: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is straightforward

Hence, we need only prove (1) $\Rightarrow$ (3). The proof is in two parts. We first show that (1) requires $\pi(A) = \tilde{\pi}(A)$ for all $A \in \mathcal{E}$, u is a concave transform of $\widehat{u}$, and then that (1) requires $b > \widehat{b} = 0$.

Part 1.

We consider acts f with the property that there exists a neighborhood $[m(f) - 2\delta, m(f) + 2\delta]$ of $m(f)$ that is not in the support of f. Call this Property 1.

For any f satisfying Property 1, we can partition S into disjoint events E (elation) and E' (the complement of E) such that

$$f(s) < m(f) - 2\delta \quad s \in E'$$
$$f(s) > m(f) + 2\delta \quad s \in E$$

Hence, for any g, $|g(s)| < \delta$, we have

$$f(s) + g(s) < m(f) - \delta \quad s \in E'$$
$$f(s) + g(s) > m(f) + \delta \quad s \in E$$

and hence

$$u(m(f+g)) - u(m(f)) = \int_{E'} u((f+g)(s)) - u(f(s))\,d\pi + \int_E (1-b)(u((f+g)(s)) - u(f(s)$$

Now by Unboundedness, we can choose f satisfying Property 1 for both $\succsim$ and $\widehat{\succsim}$ and such that $\pi(E)$ is arbitrarily small for both $\succsim$ and $\widehat{\succsim}$. Since the integrand in the second term on the RHS is bounded, this term can be made arbitrarily close to zero, in particular smaller in absolute value than any $\Delta > 0$.

Suppose for some g, $|g(s)| < \delta$, we have $m(f) = m(f+g)$. Then

$$\int_{E'} u((f+g)(s)) - u(f(s))\,d\pi + \int_E (1-b)(u((f+g)(s)) - u(f(s)))\,d\pi = 0$$

implies

$$\left|\int_S u((f+g)(s)) - u(f(s))\,d\pi\right| < \Delta$$

We can now apply the argument of the SEU proposition to show that $\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$ only if $\pi(A) = \tilde{\pi}(A)$ for all $A \in \mathcal{E}$ and u is a concave transform of $\widehat{u}$

Part 2.

It is trivial that we require $b \geq \widehat{b}$. Suppose $b \geq \widehat{b} > 0$, and that $\pi(A) = \tilde{\pi}(A)$ for all $A \in \mathcal{E}$, u is a concave transform of $\widehat{u}$. Then the Gul result shows that for any non-trivial f, $m(f) < \hat{m}(f)$. Let E, $\hat{E}$ be the elation events. By choosing f with a probability mass in the interval $[m(f), \hat{m}(f)]$, we can make the measure of $E - \hat{E}$ as large as we like relative to E' and $\hat{E}$. Choose δ and δ' to define the event g

$$g(s) = \begin{cases} \delta & s \in \hat{E} \\ -\delta' & s \in \hat{E}' \end{cases}$$

such that

$$\left(1 - \widehat{b}\right)\left(\int_{\hat{E}} (u(f+\delta) - u(f))\,d\pi\right) - \int_{\hat{E}'} (u(f) - u(f-\delta'))\,d\pi = 0$$

so that $\hat{m}(f) = \hat{m}(f+g)$. Note that this can be rewritten as

$$\left(1-\widehat{b}\right)\left(\int_{\hat{E}} (u(f+\delta) - u(f))\,d\pi\right) - \int_{E-\hat{E}} (u(f) - u(f-\delta'))\,d\pi$$
$$- \int_{E'} (u(f) - u(f-\delta'))\,d\pi = 0$$

Hence provided the measure of $E - \hat{E}$ is large enough

$$(1-b)\left(\int_{\hat{E}} (u(f+\delta) - u(f))\,d\pi\right) - (1-b)\left(\int_{E-\hat{E}} (u(f) - u(f-\delta'))\,d\pi\right)$$
$$- \int_{E'} (u(f) - u(f-\delta'))\,d\pi > 0$$

so that $m(f+g) > m(f)$ and hence $\succsim$ is not everywhere at least as uncertainty averse as $\hat{\succsim}$ ■

Proof of Proposition 8.

Necessity of the condition $P_{(\nu,\hat{\nu})} \geq G_{(u,\hat{u})}$.

Fix $E \in \mathcal{E}$, such that $\nu(E) \neq 0, 1$ and $-\infty < x^1 < x^2 \leq x^3 < \infty$. Let $x^4 = \hat{u}^{-1}\left(\hat{u}(x^3) + [\hat{u}(x^2) - \hat{u}(x^1)][1-\nu(E)]/\nu(E)\right)$. Consider the pair of acts $f = x^3_E x^2$ and $g = x^4_E x^1$. By construction, g constitutes an elementary increase in uncertainty over f. Now since

$$\begin{aligned} &\int_{-\infty}^{\infty} [\hat{\nu}(\{s : \hat{u}(f(s)) \geq w\}) - \hat{\nu}(\{s : \hat{u}(g(s)) \geq w\})]\,dw \\ = \ & [1-\hat{\nu}(E)]\left[\hat{u}\left(x^2\right) - \hat{u}\left(x^1\right)\right] - \hat{\nu}(E)\left[\hat{u}\left(x^4\right) - \hat{u}\left(x^3\right)\right] = 0 \end{aligned}$$

we have $f \mathrel{\hat{\sim}} g$. So for (u,ν) to be at least uncertainty averse, requires $f \succsim g$, or,

$$\int_{-\infty}^{\infty} [\nu(\{s : u(f(s)) \geq w\}) - \nu(\{s : u(g(s)) \geq w\})]\,dw \geq 0.$$

This is equivalent to,

$$\frac{[u(x^4) - u(x^3)]/[u(x^2) - u(x^1)]}{[\hat{u}(x^4) - \hat{u}(x^3)]/[\hat{u}(x^2) - \hat{u}(x^1)]} \leq \frac{[1-\nu(E)]/\nu(E)}{[1-\hat{\nu}(E)]/\hat{\nu}(E)}. \tag{2}$$

Hence a necessary condition for $\succsim$ to be at least as uncertainty averse as $\hat{\succsim}$ is that the supremum of the left-hand side of (2) (which is related to $G_{(u,\hat{u})}$, the index of *relative greediness* of u compared to $\hat{u}$) be less than or equal to the infimum of the right-hand side of (2), which *is* $P_{(\nu,\hat{\nu})}$, the index of *relative pessimism*. We are almost done, except that the supremum of the left-hand side of (2) is taken over vectors (x^1, x^2, x^3, x^4) satisfying $x^1 < x^2 < x^3 < x^4$ and also $x^4 = \hat{u}^{-1}(\hat{u}(x^3) + [\hat{u}(x^2) - \hat{u}(x^1)][1-\nu(E)]/\nu(E))$, so this supremum could in principle depend on $\nu(E)$. But the following lemma shows that this supremum is equal to $G_{(u,\hat{u})}$ and thus independent of $\nu(E)$.

Lemma 15 *Let u and $\hat{u}$ be continuous and increasing and let $E \in \mathcal{E}$, such that $\nu(E) \neq 0, 1$. Define*

$$\begin{aligned}
\mathbf{X}_E &= \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 : x^1 < x^2 \leq x^3 < x^4, \frac{[\hat{u}(x^4) - \hat{u}(x^3)]}{[\hat{u}(x^2) - \hat{u}(x^1)]} = \frac{[1-\nu(E)]}{\nu(E)} \right\} \\
G_{(u,\hat{u})}(E) &= \sup_{(x^1,x^2,x^3,x^4) \in \mathbf{X}_E} \left(\frac{[u(x^4) - u(x^3)] / [u(x^2) - u(x^1)]}{[\hat{u}(x^4) - \hat{u}(x^3)] / [\hat{u}(x^2) - \hat{u}(x^1)]} \right)
\end{aligned}$$

P roof. arly $G_{(u,\hat{u})}(E) \leq G_{(u,\hat{u})}$. So it remains to show $G_{(u,\hat{u})}(E) \geq G_{(u,\hat{u})}$, that is, for any $(x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ such that $x^1 < x^2 \leq x^3 < x^4$ and any $\varepsilon > 0$, there exists a vector $(y^1, y^2, y^3, y^4) \in \mathbf{X}_E$ such that

$$\frac{[u(y^4) - u(y^3)] / [u(y^2) - u(y^1)]}{[\hat{u}(y^4) - \hat{u}(y^3)] / [\hat{u}(y^2) - \hat{u}(y^1)]} \geq \frac{[u(x^4) - u(x^3)] / [u(x^2) - u(x^1)]}{[\hat{u}(x^4) - \hat{u}(x^3)] / [\hat{u}(x^2) - \hat{u}(x^1)]} - \varepsilon$$

Set $\lambda := [1-\nu(E)]/\nu(E)$. Now by continuity of u and $\hat{u}$, there exists some $x^0 \in (x^3, x^4)$ such that for every $x \in (x^0, x^4)$,

$$\frac{[u(x) - u(x^3)] / [u(x^2) - u(x^1)]}{[\hat{u}(x) - \hat{u}(x^3)] / [\hat{u}(x^2) - \hat{u}(x^1)]} \geq \frac{[u(x^4) - u(x^3)] / [u(x^2) - u(x^1)]}{[\hat{u}(x^4) - \hat{u}(x^3)] / [\hat{u}(x^2) - \hat{u}(x^1)]} - \varepsilon$$

Divide the interval $(\hat{u}(x^1), \hat{u}(x^2))$ into K sub-intervals of equal length $\Delta = (\hat{u}(x^1), \hat{u}(x^2))/K$ such that $\lambda\Delta < \hat{u}(x^4) - \hat{u}(x^0)$. This guarantees that the sequence $\hat{u}(x^3)$, $\hat{u}(x^3) + \lambda\Delta$, $\hat{u}(x^3) + 2\lambda\Delta$, $\hat{u}(x^3) + 3\lambda\Delta, \ldots$ has some element $\hat{u}(x^3) + k\lambda\Delta$ in the interval $(\hat{u}(x^0), \hat{u}(x^4))$. Let $x = \hat{u}^{-1}(\hat{u}(x^3) + k\lambda\Delta)$. Since

$$\begin{aligned}
& \left[u(x) - u\left(x^3\right)\right] / \left[\hat{u}(x) - \hat{u}\left(x^3\right)\right] \\
= \; & (1/k) \sum_{i=0}^{k-1} \left[u \circ \hat{u}^{-1}\left(\hat{u}\left(x^3\right) + (i+1)\lambda\Delta\right) - u \circ \hat{u}^{-1}\left(\hat{u}\left(x^3\right) + i\lambda\Delta\right)\right] / \lambda\Delta,
\end{aligned}$$

there is a sub-interval $(\hat{u}(y^3), \hat{u}(y^4)) = (\hat{u}(x^3) + i\lambda\Delta, \hat{u}(x^3) + (i+1)\lambda\Delta)$ of $(\hat{u}(x^3), \hat{u}(x^4))$ such that $[u(y^4) - u(y^3)] / [\hat{u}(y^4) - \hat{u}(y^3)] \geq [u(x) - u(x^3)] / [\hat{u}(x) - \hat{u}(x^3)]$. Similarly, there exists a sub-interval $(\hat{u}(y^1), \hat{u}(y^2)) = (\hat{u}(x^3) + j\Delta, \hat{u}(x^3) + (j+1)\Delta)$ of $(\hat{u}(x^1), \hat{u}(x^2))$ along which $[u(y^2) - u(y^1)] / [\hat{u}(y^2) - \hat{u}(y^1)] \geq [u(x^2) - u(x^1)] / [\hat{u}(x^2) - \hat{u}(x^1)]$. This completes the proof of the lemma. ■

Sufficiency of the condition $P_{(\nu,\hat{\nu})} \geq G_{(u,\hat{u})}$.

Fix a pair of acts, f and g, s.t. gUf. In particular, suppose for some $\alpha, \beta > 0$, some partition $\{E_1, \ldots, E_n\}$ and outcomes $x^1 \leq \ldots \leq x^n$:

$$\begin{aligned} f &= x^1_{E_1} x^2_{E_2} \ldots x^j_{E_j} x^{j+1}_{E_{j+1}} \ldots x^{n-1}_{E_{n-1}} x^n \text{ and} \\ g &= \left(x^1 - \beta\right)^2_{E_1} \left(x^2 - \beta\right)_{E_2} \ldots \left(x^\ell - \beta\right)_{E_\ell} \left(x^{\ell+1} + \alpha\right)_{E_{\ell+1}} \ldots \left(x^{n-1} + \alpha\right)_{E_{n-1}} \left(x^n + \alpha\right) \end{aligned}$$

Let $A = \bigcup_{i=\ell+1}^n E_i$.

It suffices to show that $f \hat{\sim} g$ implies $f \succsim g$. [4] If $f \hat{\sim} g$ then

$$\begin{aligned} &\sum_{i=1}^{\ell} \left[\hat{u}\left(x^i\right) - \hat{u}\left(x^i - \beta\right)\right] \left[\hat{\nu}\left(\bigcup\nolimits_{j=i}^n E_j\right) - \hat{\nu}\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] \\ &- \sum_{i=\ell+1}^{n} \left[\hat{u}\left(x^i + \alpha\right) - \hat{u}\left(x^i\right)\right] \left[\hat{\nu}\left(\bigcup\nolimits_{j=i}^n E_j\right) - \hat{\nu}\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] = 0 \end{aligned}$$

where by convention $\bigcup_{j=n+1}^n E_j := \varnothing$. By mean-value theorem, there exists $y^2 \in \left[x^1, x^\ell\right]$ and $y^3 \in \left[x^{\ell+1}, x^n\right]$ such that

$$\begin{aligned} &\sum_{i=1}^{\ell} \left[\hat{u}\left(x^i\right) - \hat{u}\left(x^i - \beta\right)\right] \left[\hat{\nu}\left(\bigcup\nolimits_{j=i}^n E_j\right) - \hat{\nu}\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] \\ = &\left[\hat{u}\left(y^2\right) - \hat{u}\left(y^2 - \beta\right)\right] \left[1 - \hat{\nu}(A)\right] \\ = &\sum_{i=\ell+1}^{n} \left[\hat{u}\left(x^i + \alpha\right) - \hat{u}\left(x^i\right)\right] \left[\hat{\nu}\left(\bigcup\nolimits_{j=i}^n E_j\right) - \hat{\nu}\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] \\ = &\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right] \hat{\nu}(A) \end{aligned}$$

Just remains to show that $f \succsim g$, or equivalently, that

$$\begin{aligned} &\sum_{i=1}^{\ell} \left[u\left(x^i\right) - u\left(x^i - \beta\right)\right] \left[\nu\left(\bigcup\nolimits_{j=i}^n E_j\right) - \nu\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] \\ &- \sum_{i=\ell+1}^{n} \left[u\left(x^i + \alpha\right) - u\left(x^i\right)\right] \left[\nu\left(\bigcup\nolimits_{j=i}^n E_j\right) - \nu\left(\bigcup\nolimits_{j=i+1}^n E_j\right)\right] \geq 0 \end{aligned}$$

Now let m and m' be the indices satisfying

$$\begin{aligned} \left[u\left(x^m\right) - u\left(x^m - \beta\right)\right] &\leq \left[u\left(x^i\right) - u\left(x^i - \beta\right)\right], \text{ for } i = 1, \ldots, \ell \\ \left[u\left(x^{m'} + \alpha\right) - u\left(x^{m'}\right)\right] &\geq \left[u\left(x^i + \alpha\right) - u\left(x^i\right)\right], \text{ for } i = \ell, \ldots, n \end{aligned}$$

[4] If $f \hat{\succ} g$, then there exists $\varepsilon > 0$, such that the act f', defined by $f'(s) = f(s) - \varepsilon$, we have $f' \hat{\sim} g$. Hence if $f' \hat{\sim} g$ implies $f' \succsim g$, by monotonicity we have $f \succ f'$ and thus by transitivity, $f \succ g$.

Hence

$$\sum_{i=1}^{\ell} \left[u\left(x^i\right) - u\left(x^i - \beta\right)\right] \left[\nu\left(\bigcup_{j=i}^{n} E_j\right) - \nu\left(\bigcup_{j=i+1}^{n} E_j\right)\right]$$
$$- \sum_{i=\ell+1}^{n} \left[u\left(x^i + \alpha\right) - u\left(x^i\right)\right] \left[\nu\left(\bigcup_{j=i}^{n} E_j\right) - \nu\left(\bigcup_{j=i+1}^{n} E_j\right)\right]$$
$$\geq \left[u\left(x^m\right) - u\left(x^m - \beta\right)\right]\left[1 - \nu\left(A\right)\right] - \left[u\left(x^{m'} + \alpha\right) - u\left(x^{m'}\right)\right]\nu\left(A\right)$$
$$= \left(\frac{\left[u\left(x^m\right) - u\left(x^m - \beta\right)\right]\left[1 - \nu\left(A\right)\right]}{\left[\hat{u}\left(y^2\right) - \hat{u}\left(y^2 - \beta\right)\right]\left[1 - \hat{\nu}\left(A\right)\right]} - \frac{\left[u\left(x^{m'} + \alpha\right) - u\left(x^{m'}\right)\right]\nu\left(A\right)}{\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right]\hat{\nu}\left(A\right)}\right)\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right] i$$
$$= \frac{\left[u\left(x^m\right) - u\left(x^m - \beta\right)\right]\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right]\nu\left(A\right)}{\left[\hat{u}\left(y^2\right) - \hat{u}\left(y^2 - \beta\right)\right]}$$
$$\times \left(\frac{\left[1 - \nu\left(A\right)\right]/\nu\left(A\right)}{\left[1 - \hat{\nu}\left(A\right)\right]/\hat{\nu}\left(A\right)} - \frac{\left[u\left(x^{m'} + \alpha\right) - u\left(x^{m'}\right)\right]/\left[u\left(x^m\right) - u\left(x^m - \beta\right)\right]}{\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right]/\left[\hat{u}\left(y^2\right) - \hat{u}\left(y^2 - \beta\right)\right]}\right)$$
$$\geq \frac{\left[u\left(x^m\right) - u\left(x^m - \beta\right)\right]\left[\hat{u}\left(y^3 + \alpha\right) - \hat{u}\left(y^3\right)\right]\nu\left(A\right)}{\left[\hat{u}\left(y^2\right) - \hat{u}\left(y^2 - \beta\right)\right]}\left(P_{(\nu,\hat{\nu})} - G_{(u,\hat{u})}\right)$$

But $P_{(\nu,\hat{\nu})} \geq G_{(u,\hat{u})}$ and thus $f \succsim g$, as required. ■

Proof of Proposition 12 Fix two comonotonic acts f and g.

We first consider the case in which the two acts exhibit a "finite-crossing" property in the sense that there exists a finite partition, $\left\{E^1, \ldots, E^K\right\}$ that is ordered with respect to f (and hence also with respect to g) and such that, for each k, either

$$(a) \quad g(s) \geq f(s) \quad \forall s \in E^k$$

or

$$(b) \quad g(s) \leq f(s) \quad \forall s \in E^k$$

If (a) holds on E^K and (b) holds on E^1, then the standard analysis under risk applies - note that by selecting the probability distribution over states appropriately, we can always ensure that g is more risky than f in the sense of Rothschild-Stiglitz. That is, there exists a probability distribution $\mu\left(.\right)$ defined over $\mathcal{E}$ such that for all x

$$\int_0^x \left(\mu\left(\left\{s : f\left(s\right) \leq y\right\}\right) - \mu\left(\left\{s : g\left(s\right) \leq y\right\}\right)\right) dy \leq 0$$

and

$$\int_0^\infty \left(\mu\left(\{s : f(s) \leq y\}\right) - \mu\left(\{s : g(s) \leq y\}\right)\right) dy = 0$$

Hence, as Machina and Pratt (1997) show there exists a sequence of simple mean preserving spreads

Consider the case where (a) holds on E^1 and E^K. (note that this includes the case when $g(s) \geq f(s) \quad \forall s$). Now consider any descending series of non-empty sets $E^1 = A^1 \supset A^2$.... such that

(i)

$$\bigcap_{i=1}^{\infty} A^i = \emptyset$$

(ii)

$$\sup\left\{f(s) : s \in A^i\right\} \leq \inf\left\{f(s) : s \in E^1 \backslash A^i\right\}$$

Small-event continuity ensures that such a series exists. Now consider acts h^i such that for some $\delta > 0$.

$$h^i(s) = \begin{cases} g(s) & s \in \mathcal{S} \backslash A^i \\ f(s) + \delta & s \in A^i \end{cases}$$

Then, by the argument already given, h^i is in the transitive closure of the elementary transfer relation, and $h^i \to g$. ■

References

Arrow, K. and Debreu, G. (1954), "Existence of an equilibrium for a competitive economy," *Econometrica*, **22**, 265-90.

Bartik, T. (1988), 'Evaluating the benefits of non-marginal reductions in pollution using information on defensive expenditures', *Journal of Environmental Economics and Management* **15**, 111–127.

Bickel, P.J. and Lehmann, E.L. (1976), "Descriptive statistics for non-parametric models, III. Dispersion." *Annals of Statistics*, **4**, 1139-58.

Chambers, R. G. and Quiggin, J. (2000), *Uncertainty, Production, Choice and Agency: The State-Contingent Approach*, Cambridge University Press, New York.

Chateauneuf, A. and Cohen, M. (1994), "Risk-seeking with diminishing marginal utility in a non-expected utility model," *Journal of Risk and Uncertainty,* **9**, 77-91.

Chateauneuf, A.,Cohen, M., and Meilijson, I. (1997), "New tools to better model behavior under risk and uncertainty: An overview," *Finance*, **18**, 25-46.

Chateauneuf, A.,Cohen, M., and Meilijson, I. (2004a), "More pessimism than greediness: a characterization of monotone risk aversion in the rank-dependent expected utility model," forthcoming *Economic Theory.*

Chateauneuf, A.,Cohen, M., and Meilijson, I. (2004b), "Four notions of mean preserving increase in risk, risk attitudes and applications to the Rank-Dependent Expected Utility model, *Journal of Mathematical Economics*, **40**(5), 547-571.

Chew, S-H. (1983), "A generalization of the quasi-linear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox," *Econometrica*, **51**, 1065-92.

Chew, S-H., Karni, E., and Safra, Z. (1987), "Risk aversion in the theory of expected utility with rank dependent preferences," *Journal of Economic Theory*, **42**, 370-81.

Chew, S-H and Wakker, P. (1996), "The comonotonic sure-thing principle," *Journal of Risk and Uncertainty*, **12**, 5-27.

de Finetti, B. (1937), "La prévision: ses lois logiques, ses sources subjectives," *Annales de l'Institut Henri Poncaré*, **7**, 1-68. (Translated in Kyburg and Smokler, 1964).

Diamond, P. A. and Stiglitz, J. E. (1974), "Increases in risk and risk aversion," *Journal of Economic Theory*, **8**(3), 337-60.

Ellsberg, D. (1961), "Risk, ambiguity and the Savage axioms," *Quarterly Journal of Economics*, **75**(4), 643-69.

Epstein, L. G. (1999), "A definition of uncertainty aversion," *Review of Economic Studies*, **66**(3), 579-608.

Epstein, L. G. and Zhang, J. (2001), "Subjective probabilities on subjectively unambiguous events," *Econometrica*, **69**(2), 265-306.

Feder, G. (1977), 'The impact of uncertainty in a class of objective functions', *Journal of Economic Theory*, **16**, 504—12.

Ghirardato, P. and Marinacci, M. (2002), "Ambiguity made precise: a comparative foundation," *Journal of Economic Theory*, **102**, 251-289. (doi:10.1006/jeth.2001.2815)

Ghirardato, P., Maccheroni, F. and Marinacci, M. (2004), "Differentiating ambiguity and ambiguity attitude," forthcoming *Journal of Economic Theory.*

Gilboa, I. and Schmeidler, D. (1994), "Updating ambiguous beliefs," *Journal of Economic Theory*, **59**, 33-49.

Grant, S., Kajii, A. and Polak, B. (1992), "Many good risks: an interpretation of multivariate risk and risk aversion without the independence axiom" *Journal of Economic Theory*, **56**, 338-351.

Hadar, J. and Russell, W. (1969), "Rules for ordering uncertain prospects," *American Economic Review*, **59**(1), 25-34.

Hanoch, G. and Levy, H. (1969), "The efficiency analysis of choices involving risk," *Review of Economic Studies*, **36**, 335-346.

Knight, F. (1921), *Risk, Uncertainty and Profit*, Houghton Mifflin, New York.

Kyburg, H.E. Jr., and Smokler, H.E. (1964), *Studies in Subjective Probability*, New York, John Wiley and Sons.

Landsberger, M. and Meilijson, I. (1994), "Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion," *Annals of Operations Research*, **52**, 97-106.

Machina, M. J.and Pratt, J. W. (1997), "Increasing Risk: Some Direct Constructions", *Journal of Risk and Uncertainty*, **14** (2), 103-27.

Machina, M. J. and Schmeidler, D. (1992), "A more robust definition of subjective probability," *Econometrica*, **60** (4), 745-80.

Mukerji, S. (1997), "Understanding the nonadditive probability decision model," *Economic Theory*, **9**, 23-46.

Nehring, K. (1999), "Capacities and probabilistic beliefs: a precarious coexistence," *Mathematical Social Sciences*, **38**, 197-213.

Nehring, K. (2001), "Ambiguity in the Context of Probabilistic Beliefs," mimeo University of California, Davis.

Pratt, J. W. (1964), "Risk aversion in the small and the large," *Econometrica*, **32**, 122-36.

Quiggin, J. (1991), 'Comparative statics for Rank-Dependent Expected Utility theory', *Journal of Risk and Uncertainty*, **4**(4), 339–50.

Quiggin, J. (1993), *Generalized Expected Utility Theory: The Rank-Dependent Model*, Kluwer, Amsterdam.

Rothschild, M. and Stiglitz, J. (1970), "Increasing risk: I. A definition," *Journal of Economic Theory*, **2**(4), 225-243.

Ryan, M. (2002), "What do uncertainty-averse decision-makers believe?", *Economic Theory*, **20**(1), 47-65.

Safra, Z. and Zilcha, I. (1989), "Efficient sets with and without the expected utility hypothesis," *Journal of Mathematical Economics*, **17**, 369-84.

Sarin, R. and Wakker, P. (1998), "On the interpretation of likelihood in Choquet Expected Utility," *Journal of Risk and Uncertainty*, **16**, 223-50.

Savage, L. (1954), *The Foundations of Statistics.* New York Wiley (Second Edition, 1972, Dover).

Schmeidler, D. (1989), "Subjective probability and expected utility without additivity," *Econometrica*, **57**, 571-87.

Schmidt, U. and H. Zank (2002), "Risk aversion in cumulative prospect theory," mimeo, School of Economic Studies, The University of Manchester.

Segal, U. (1989), "Anticipated utility: A measure representation approach," *Annals of Operations Research*, **19** , 359-73.

Yaari, M. (1969), "Some remarks on measures of risk aversion and their uses," *Journal of Economic Theory*, **1**(3), 315-29.

Yaari, M. (1987), "The dual theory of choice under risk," *Econometrica*, **55**, 95-115.