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Capital market equilibrium with moral hazard and flexible technology

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Abstract

Magill and Quinzii (2002) show that, in a stockmarket economy with private information, the moral hazard problem may be resolved provided that a spanning overlap condition is satisfed. This result depends on the assumption that the technology is given by a stochastic production function with a single scalar input. The object of the present paper is to extend the analysis of Magill and Quinzii to the case of multiple inputs. We show that their main result extends to this general case if and only if, for each firm, the number of linearly independent combinations of securities having payoffs correlated with, but not dependent on, the firms output is equal to the number of degrees of freedom in the firm's production technology.

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Magill and Quinzii (2002) show that, in a stockmarket economy with private information, any state-contingent equilibrium may be generated as a financial market equilibrium provided that a spanning overlap condition is satisfed. Thus, with an appropriate specification of the security structure, the market can avoid the conflict between risk sharing and incentives that is typical of the moral hazard problem.

One notable feature of the analysis of Magill and Quinzii relates to the production technology, which is assumed to be characterized by a stochastic production function, with a single scalar input. As observed by Holmstrom and Milgrom (1987), in a standard principal agent problem, the greater the flexibility available to the agent, the more difficult the problem faced by the principal. For a highly flexible technology, Holmstrom and Milgrom show that the principal can do no better than to offer a payment schedule that is an affine function of output.

The object of the present paper is to extend the analysis of Magill and Quinzii to the case of a stochastic production function with multiple inputs. We show that their main result extends to this general case if and only if, for each firm, the number of linearly independent combinations of securities having payoffs correlated with, but not dependent on, the firms output is equal to the number of degrees of freedom in the firm's production technology. The scalar-input stochastic production function technology examined by Magill and Quinzii is the case where the firm has a single degree of freedom. In the other polar case, where the firm has enough independent inputs to span the state space, the first-best can be achieved if and only if the state space is also spanned by 'outside' securities, independent of the firm's observed output.

1 Model

Magill and Quinzii (2002) consider a simple two-period one-good economy with production. There are two types of agents: entrepreneurs and investors. $\mathcal{I}_1 \neq \emptyset$ is the set of entrepreneurs, $\mathcal{I}_2 \neq \emptyset$ is the set of investors, and the set $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ of all agents is finite. Every agent $i \in \mathcal{I}$ has an initial wealth ω_0^i at date 0. An agent i who is an entrepreneur has the opportunity to create a productive venture by investing an amount of capital κ^i. We assume that there is a finite set $\mathcal{S}$ of states of nature describing the shocks to which firms can be subjected.

State-contingent output for agent i is given by a stochastic production function

$$\mathbf{y}^i = \mathbf{F}^i\left(\kappa^i, \mathbf{e}^i\right) = \left(F_1^i\left(\kappa^i, \mathbf{e}^i\right), ..., F_S^i\left(\kappa^i, \mathbf{e}^i\right)\right),$$

where κ^i is a scalar capital input and $\mathbf{e}^i \in \Re_+^M$ is a vector of inputs that we refer to as effort, and each of the F_s^i is concave in effort. This is a generalization of the technology in Magill and Quinzii (2002), who restrict effort to be a scalar[1].

We assume that J contracts are traded, contract j being characterized by a state independent function $V^j : \Re^I \rightarrow \Re^I$ describing the way the payoff of contract j depends on the realized output of the I firms in the economy. The first I securities are assumed to denote equity shares in the I firms, with security i having payoff vector $\mathbf{y}^i$. For each i, the set of securities J can be partitioned into

$$J = J^i \cup J_{-i}^i \cup J_{-i}.$$

where J^i consists of securities for which the payoff depends exclusively on the output of firm i (including the equity of firm i and any derivative securities such as options), J_{-i}^i consists of securities for which the payoff depends on the output of firm i and the output of at least one other firm (such as index securities) and J_{-i} consists of securities for which the payoff does not depend on the output of firm i

Let $\mathbf{y} = (\mathbf{y}^1, ..., \mathbf{y}^I)$ denote the random outputs of the firms and let $\mathbf{y}_s = (y_s^1, ..., y_s^I)$ denote the realized outputs in state s. The payoff of security j in state s is then $V^j(\mathbf{y}_s)$. We let $V^j(\mathbf{y})$ denote the vector $(V^j(\mathbf{y}_s))$ $s \in \mathcal{S}$ and $\mathbf{V}(\mathbf{y}) = [V^1(\mathbf{y}), ..., V^J(\mathbf{y})]$ denote the matrix of payoffs of the J securities. The security price vector is $\mathbf{q} \in \Re^J$.

Each agent i chooses a portfolio $\mathbf{z}^i \in \Re^J$ and consumes $\mathbf{x}^i = (x_0^i, x_{\mathbf{1}}^i) = (x_0^i, x_1^i ... x_s^i)$ where

$$\begin{aligned} x_0^i &= \omega_0^i + q_i - \mathbf{q}\mathbf{z}^i - \kappa^i \\ x_{\mathbf{1}}^i &= V(\mathbf{y})\mathbf{z}^i. \end{aligned}$$

Utility for agent i is given by the function $U^i\left(\mathbf{x}^i, e^i\right)$. Following Magill and Quinzii (2002), we will assume that U is time-separable. In addition, we will simplify analysis by assuming that U is additively separable in $\mathbf{x}^i$ and e^i. That is,

$$\begin{aligned} U^i\left(\mathbf{x}^i, e^i\right) &= u_0^i\left(x_0^i\right) + u_1^i\left(\mathbf{x}_{\mathbf{1}}^i, \mathbf{e}^i\right) \\ &= u_0^i\left(x_0^i\right) + v_1^i\left(\mathbf{x}_{\mathbf{1}}^i\right) - g^i\left(\mathbf{e}^i\right), \end{aligned}$$

[1] The assumption that effort is a scalar is not made explicit until p. 171, but is implicit in the discussion throughout.

where g^i is a convex function representing disutility of effort.

2 The effort-cost function

Define the *effort-cost function for* $\mathbf{y}$ by

$$\begin{aligned}\tilde{c}^i\left(\kappa^i,\mathbf{y}\right) &= \inf\left\{g^i\left(\mathbf{e}\right):\mathbf{y}\leq\mathbf{F}^i\left(\kappa^i,\mathbf{e}\right)\right\}. \\ &= \inf\left\{g^i\left(\mathbf{e}\right):y_s\leq F_s^i\left(\kappa^i,\mathbf{e}\right),s=1...S\right\}.\end{aligned}$$

$\tilde{c}^i\left(\kappa^i,\mathbf{y}\right)$ is nondecreasing and convex in y. Convexity follows from the the assumed convexity of g^i and the fact that concavity of the stochastic production function implies that the κ^i-conditional graph of the technology, $\left\{(\mathbf{e},\mathbf{y}):\mathbf{y}\leq\mathbf{F}^i\left(\kappa^i,\mathbf{e}\right)\right\}$ is convex (Chambers and Quiggin, 2000, Result 4.1). We assume throughout that $\tilde{c}^i\left(\kappa^i,\mathbf{y}\right)$ is proper in the sense that $\tilde{c}^i\left(\kappa^i,\mathbf{y}\right)<\infty$ for at least one $\mathbf{y}$ and $\tilde{c}^i\left(\kappa^i,\mathbf{y}\right)>-\infty$ for all $\mathbf{y}$.

To apply the first-order approach it is desirable to derive a differentiable representation of the technology. We define the efficient set for i, given κ^i :

$$Y^i\left(\kappa^i\right)=\left\{\mathbf{y}\in\Re_+^S:\mathbf{y}'>\mathbf{y}\Rightarrow\tilde{c}^i\left(\kappa^i,\mathbf{y}'\right)>\tilde{c}^i\left(\kappa^i,\mathbf{y}\right)\right\}$$

where $\mathbf{y}'>\mathbf{y}$ means $y_s'\geq y_s\forall s$, with strict inequality for at least one s. That is, if $\mathbf{y}$ is in the efficient set, it is not possible to increase output in any state without also increasing the minimal cost of effort. Now, we define

Definition 1 *Let* $\Theta^i\subseteq\Re_+^{N_i}$ *be a strictly convex set with non-empty interior, including* $\mathbf{0}$. $\boldsymbol{\theta}^i\in\Theta^i$ *is a sufficient statistic for* $Y^i\left(\kappa^i\right)$, *given* κ^i, *if there exists a convex, increasing 1-1 mapping* $f^i:\Theta^i\times\Re_+\rightarrow\Re_+^S$, $f^i\left(\boldsymbol{\theta}^i;\kappa^i\right)=\mathbf{y}^i$ *such that*

$$im\left(\boldsymbol{\theta}^i;\kappa^i\right)=Y^i\left(\kappa^i\right),$$

where

$$im\left(\boldsymbol{\theta}^i;\kappa^i\right)=\left\{f^i\left(\boldsymbol{\theta}^i;\kappa^i\right):\boldsymbol{\theta}^i\in\Theta^i\right\}.$$

Given this definition, the *effort-cost function for* θ can be defined by

$$c^i\left(\boldsymbol{\theta}^i;\kappa^i\right)=\tilde{c}^i\left(\kappa^i;f^i\left(\boldsymbol{\theta}^i;\kappa^i\right)\right).$$

By the definition of a sufficient statistic and standard results on nondecreasing convex functions (Rockafellar 1970), we have

Lemma 2 *If* $\boldsymbol{\theta}^i\in\Theta^i$ *is a sufficient statistic for* $Y^i\left(\kappa^i\right),c^i\left(\boldsymbol{\theta}^i;\kappa^i\right)$ *is convex in* $\boldsymbol{\theta}^i$.

We will assume that there exists, for each i, a sufficient statistic $\theta^i \in \Re_+^{N_i}$ and an associated convex function $c^i\left(\boldsymbol{\theta}^i;\kappa^i\right)$. Because $c^i\left(\boldsymbol{\theta}^i;\kappa^i\right)$ is convex, it is differentiable almost everywhere in the interior of its domain (Rockefellar 1970). Therefore, with little true loss of generality, we shall treat it as differentiable in what follows.[2]

The dimension N_i is referred to as the number of *degrees of freedom* available to firm i. The agent's optimal production decisions, given a choice of capital input κ^i and portfolio $\mathbf{z}^i$, and conditional on the actions $\mathbf{y}^{-i}$ are given by

$$\mathbf{y}^i\left(\tilde{\boldsymbol{\theta}}^i;\kappa^i,\mathbf{z}^i;\mathbf{y}^{-i}\right) = \arg\max\left\{v_1^i\left(\mathbf{x}_1^i\right) - c^i\left(\tilde{\boldsymbol{\theta}}^i;\kappa^i\right) : x_{\mathbf{1}}^i = V\left(f^i\left(\tilde{\boldsymbol{\theta}}^i;\kappa^i\right),\mathbf{y}^{-i}\right)\mathbf{z}^i\right\}.$$

We assume there exists a well-defined maximum, not necessarily unique.

3 Equilibrium conditions

Magill and Quinzii (2002) define a number of equilibrium conditions. Two are crucial for the purposes of the present paper. These are artificial sole proprietorship (ASP) and rational competitive price perceptions (RCPP) equilbrium conditions. The notion of ASP equilibrium used here is the same as that used by Magill and Quinzii (2002), except that we explicitly include the sufficient statistic $\boldsymbol{\theta}^i$ in the description of the choices made by agent i. The notion of weak RCPP equilibrium requires some slight adjustments to take account of a technology with multiple inputs.

As in Magill and Quinzii (2002), we assume that the payoff functions $V^j : \Re_+^I \rightarrow \Re$ can have points of non-differentiability but have well-defined left and right derivatives $\left(\partial V^j/\partial y_-^i\right)$ and $\left(\partial V^j/\partial y_-^+\right)$ everywhere on $\Re_+^I$. Let

$$\pi_s^i\left(\mathbf{x}^i\right) = \frac{\partial u_1^i\left(\mathbf{x}_1^i,\mathbf{e}^i\right)/\partial x_s^i}{du_0^i\left(x_0^i\right)/\partial x_0^i} = \frac{\partial v_1^i\left(\mathbf{x}_1^i\right)/\partial x_s^i}{du_0^i\left(x_0^i\right)/\partial x_0^i}$$

denote the present value (to agent i) of an additional unit of income in state s. Now, for points of differentiability, let $b_{jn}^i\left(\mathbf{x}^i,\kappa^i,e^i,\theta^i,\mathbf{y}^{-i}\right)$ denote the derivative of the present value of security $j's$ payoff with respect to θ_n^i

$$b_{jn}^i\left(\mathbf{x}^i,\kappa^i,e^i,\boldsymbol{\theta}^i,\mathbf{y}^{-i}\right) = \frac{\partial}{\partial\theta_n^i}\sum_s \pi_s^i\left(\mathbf{x}^i\right)V^j\left(f_s^i\left(\theta_i;\kappa^i\right),\mathbf{y}_s^{-i}\right)$$

[2]Because it is also proper, c^i possesses directional derivatives and subdifferentials wherever it is finite. We can always generalize our following arguments using subdifferentials.

with b^i_{jn-} and b^i_{jn+} being defined with respect to the left and right derivatives respectively.

Define the vectors of derivatives $\mathbf{b}^i_n$, $\mathbf{b}^i_{n-}$ and $\mathbf{b}^i_{n+}$ and write the marginal cost of θ^i_n , in present value terms:

$$\gamma^i_n\left(\mathbf{x}^i, e^i, \boldsymbol{\theta}^i\right) = \frac{1}{du^i_0\left(\bar{x}^i_0\right)/\partial x^i_0}\frac{\partial c^i\left(\boldsymbol{\theta}^i; \kappa^i\right)}{\partial \theta^i_n}$$

Then, the first-order condition used in the budget set definition for a weak RCPP equilibrium becomes

$$b^i_{jn-}\left(\mathbf{x}^i, \kappa^i, e^i, \boldsymbol{\theta}^i, \mathbf{y}^{-i}\right) z^i \geq \gamma^i_n\left(\mathbf{x}^i, e^i, \boldsymbol{\theta}^i\right) \geq b^i_{jn+}\left(\mathbf{x}^i, \kappa^i, e^i, \boldsymbol{\theta}^i, \mathbf{y}^{-i}\right) z^i$$

4 The main result

Proposition 1: Let $\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \overline{\boldsymbol{\theta}}, \bar{e}, \bar{\kappa}; \bar{\boldsymbol{\pi}}\right)$ be an ASP equilibrium. If V is a security structure based on the observable outputs of the firms, which is differentiable at $\bar{\mathbf{y}}$ and satisfies

(i) (SPANNING) Rank $V\left(\bar{\mathbf{y}}\right) = S$

(ii) (OVERLAP) For each $i \in I_1$, there exists a set of N^i linearly independent outside income streams $\mathbf{v}^i_n \in V_{-i}\left(\mathbf{y}\right)$ and a matrix of coefficients λ^i_{jn} for $j \in J^i \cup J^i_{-i}$, with rank N^i, such that, for each n,

$$\begin{aligned} \mathbf{v}^i_n &= \sum_{j \in J^i \cup J^i_{-i}} V^j\left(\bar{\mathbf{y}}\right)\lambda^i_{jn}, \text{ and} \\ \sum_{j \in J^i \cup J^i_{-i}} b^i_{jn}\lambda^i_{jn} &\neq 0. \end{aligned}$$

Then there exists portfolios $\bar{\mathbf{z}}$ such that $\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \overline{\boldsymbol{\theta}}, \bar{e}, \bar{\kappa}, \bar{\mathbf{z}}; \bar{\boldsymbol{\pi}}\right)$ is a weak-RCPP equilibrium.

The proof, which is a straightforward adaption of that provided by Magill and Quinzii (2002) is presented in an Appendix.

5 Discussion

The significance of Proposition 1 may be appreciated by considering the polar cases $N^i = 1$ and $N^i = S$. The result for $N^i = 1$ includes, as a special case, that analyzed by Magill and Quinzii (2002). More generally, $N^i = 1$ globally if and only if the technology has the effort inputs separable from the capital input. That is, there exists $g^i : \Re_+^M \to \Re_+$, $G^i : \Re_+^2 \to \Re_+^S$, such that:

$$\mathbf{F}^i\left(\kappa^i, \mathbf{e}^i\right) = G^i\left(\kappa^i, \theta^i\left(\mathbf{e}^i\right)\right).$$

Under these conditions, the scalar sufficient statistic $\theta^i\left(\mathbf{e}^i\right)$ may be referred to as a composite input.

In the other polar case, $N^i = S$, the conditions of Proposition 1 are met only if, for each firm i, the outside securities span the state space. This implies that, for each i, the state of nature can be inferred from securities payoffs without reference to firm i. Hence, it is possible to implement a payment rule, conditional on the sufficient statistic $\theta^i\left(\mathbf{e}^i\right)$, using only information on outside securities.

Now consider cases under which the conditions of Proposition 1 are not met. In general, under these circumstances, firms will have welfare-relevant private information about the state of nature, and their own choice of effort vector, as summarized by the sufficient statistic θ^i, that cannot be inferred from payoffs to outside securities. Under these circumstances, as shown by Quiggin and Chambers (1998) and Chambers and Quiggin (2000), the optimal strategy for firms will involve a mixture of self-protection (the adoption of a less risky production strategy than in ASP equilibrium) and self-insurance (bearing some of the uninsurable idiosyncratic risk).

6 Concluding comments

When state-contingent output is a function of a single scalar input, Magill and Quinzii (2002) the existence of a single outside security appropriately correlated with the output of the firm is sufficient to resolve the problem arising from the fact that holders of securities issued by the firm cannot monitor the effort input of the firm's owner. In this paper, we have shown that, the more degrees of freedom the firm has available, the greater the number of linearly independent outside securities required to resolve the moral hazard problem. In most cases, it seems reasonable to suppose that the existence of outside securities will mitigate, but not eliminate, the moral hazard problem.

7 References

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8 Appendix

Proof of Proposition 1: The proof of Magill and Quinzii (2002) goes through unchanged, except for the component proving that, for each $i \in I_1$, the consumption bundle $\bar{\mathbf{x}}_1^i$ lies in the constrained budget set $B'\left(\pi, \omega_0^i, T^i, \bar{\mathbf{y}}^i\right)$. We begin by choosing a portfolio $\tilde{\mathbf{z}}^i$ such that $\mathbf{x}_1^i = \mathbf{V}\left(\bar{\mathbf{y}}\right)\tilde{\mathbf{z}}^i$; this exists since $V\left(\bar{\mathbf{y}}\right) = S$. Now decompose $\bar{\mathbf{x}}_1^i$ into i-dependent and i-independent subspaces

$$\begin{aligned}
\bar{\mathbf{x}}_1^i &= \sum_{j \in J_{-i}} V^j\left(\bar{\mathbf{y}}\right)\tilde{z}_j^i + \sum_{j \in J^i \cup J_{-i}^i} V^j\left(\bar{\mathbf{y}}\right)\tilde{z}_j^i \\
&= \sum_{j \in J_{-i}} V^j\left(\bar{\mathbf{y}}\right)\tilde{z}_j^i - \sum_n \rho_n \mathbf{v}_n^i + \sum_{j \in J^i \cup J_{-i}^i} V^j\left(\bar{\mathbf{y}}\right)\left(\tilde{z}_j^i + \sum_n \rho_n \lambda_{jn}^i\right)
\end{aligned}$$

for any vector $\boldsymbol{\rho} \in \Re^N$. Since $\mathbf{v}_n^i \in V_{-i}\left(\bar{\mathbf{y}}\right)$, for all n, there exist $\left(\mu_{jn}^i\right)$ such that $\mathbf{v}_n^i = \sum_{j \in J_{-i}} V^j\left(\bar{\mathbf{y}}\right)\mu_{jn}^i$. Moreover, by linear independence of the $\mathbf{v}_n^i$, the matrix with j,n entry μ_{jn}^i has rank N. Thus, for all $\boldsymbol{\rho} \in \Re^N$, there exists a portfolio $\mathbf{z}^{i\square}$ with

$$z^{i\square} = \begin{array}{ll} \tilde{z}_j^i - \sum_n \rho_n \mu_{jn}^i & j \in J_{-i} \\ \tilde{z}_j^i + \sum_n \rho_n \lambda_{jn}^i & j \in J^i \cup J_{-i}^i \end{array}$$

which leads to the same consumption stream $\bar{\mathbf{x}}_1^i$. Since $\sum_{j \in J^i \cup J_{-i}^i} b_{jn}^i \lambda_{jn}^i \neq 0$ and the matrix of coefficients λ is of rank N, there exists $\bar{\boldsymbol{\rho}} \in \Re^N$ such that, for all n,

$$\bar{\mathbf{b}}_n^i z^{i\square} = \sum_{j \in J^i \cup J_{-i}^i} b_{jn}^i \tilde{z}_j^i + \bar{\rho}_n \left(\sum_{j \in J^i \cup J_{-i}^i} b_{jn}^i \lambda_{jn}^i\right) = \gamma_n^i\left(\overline{\mathbf{x}}^i, \bar{e}^i, \overline{\boldsymbol{\theta}}^i\right)$$

and the remainder of the Magill and Quinzii (2002) proof goes through.