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Information and the Risk-Averse Firm

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1 Introduction

Sandmo (1971) developed a path-breaking extension of the portfolio selection problem in considering the production choices of a risk-averse, expected-utility-of-net-returns maximizing firm facing price but not production uncertainty. Extensions of the Sandmovian model to encompass production uncertainty have been developed by Newbery and Stiglitz (1981) and a host of subsequent writers. In recent years, the value of information for producing firms facing either price or production uncertainty in the Sandmovian framework has become a particular focus of analysis (Lehmann, 1988; Ormiston and Schlee (1992, 1993); Gollier, 1995; Eeckhoudt and Gollier, 1995; Athey, 2002; Athey and Levin, 2000; and Athey, 2000). In common with most analyses of production under uncertainty, however, many of these studies focus on the case of a scalar decision variable, effort, that shifts a stochastic production function. The resulting analysis is very mathematically elegant, but it is also highly specialized. And in most instances, the analysis bears little or no resemblance to the types of arguments encountered in other areas of production economics.

This paper has two goals. First, it seeks to demonstrate that standard arguments and methods from production and duality analysis can be used to provide a comprehensive and general treatment of the value of information for a risk-averse firm with expected-utility (linear-in-probabilities) preferences and a general stochastic technology. Hence, one focus is on integrating the analysis of informational-value problems into a framework commonly used (and understood) by production analysts. Our central observation is that decision making for individuals with linear-in-probabilities preferences is equivalent to decision-making over closed, convex sets. While this observation is familiar and, by now, trivial in modern production analysis, its consequences for risk-averse firms and, in particular, for studying the value of information to such firms do not seem to have yet been appreciated. But, in fact, this simple realization permits examination of the value of information and the impact of information provision in their most natural terms, which is the space of probability distributions, much in the same manner that one obtains an exhaustive examination of the theory of the price-taking firm in the natural terms of price space.

The second goal of this paper is to use this set up to place bounds on the value of information for a risk-averse firm and to relate these bounds to characteristics of the technology and the producer's preferences. A particularly striking observation that emerges from this representation is that the most common representation of production uncertainty corresponds to a polar case that trivializes the role that information can play in economic decisionmaking under risk.

In what follows, we first introduce our model which includes the specification of an event-contingent production technology and producer preferences over stochastic outcomes. We use those specifications to deduce the properties of a utility correspondence induced by the composition of the two. We then show that it can be appropriately convexified and used as the basis for the study of decisionmaking. An immediate consequence of this convexification is a general method for computing the value of information to a risk-averse firm, which is applicable for both multidimensional and scalar decision variables and which affords easy calculation of bounds on the value of information and an easy method for determining when information is valueless. Bounds are calculated and discussed. The paper then concludes.

2 Technology and Information in a State-Contingent Framework

2.1 The state space

Uncertainty is modelled by a neutral player, Nature, making a choice from a finite set, Ω of the form $\Omega = S \times N$, which allows for S possible events relevant to production and N possible signals. A signal $n \in N$, typically, will be taken to correspond to a partition of Ω according to the events $\{1_n, ..., S_n\}$. More generally, one can think of this partition into N signals as the 'finest' possible partition of the state space. Coarser partitions of the state space can be derived by considering the remaining elements of the power set of $\{1, 2, ..., N\}$. The entire set of partitions can be ordered by inclusion.

2.2 Event-contingent production

Following Chambers and Quiggin (2000), the stochastic technology, which is the same for all signals,[1] is represented by an event-contingent output correspondence. Let $\mathbf{x} \in \Re_+^M$ be a vector of inputs committed prior to the resolution of uncertainty, that is, prior to Nature's choice, and let $\mathbf{z} \in \Re_+^S$ be a vector of event-contingent outputs also chosen prior to the realization of s. Thus, if event s is realized, output z_s is produced. We confine attention to the case of a scalar output. The technology is characterized by the *event-contingent output correspondence*, which gives the vectors of event-contingent outputs that can be produced by a given vector of inputs. Formally, it is defined by:

$$Z(\mathbf{x}) = \{\mathbf{z} \in \Re_+^S : \mathbf{x} \quad \text{can} \quad \text{produce} \quad \mathbf{z}\}.$$

The image of the output correspondence is referred to as the *output set.*

Several examples illustrate. The most familiar, the simple portfolio-selection problem, involves no physical production. Let w be an individual's wealth and suppose that she can allocate it between a safe asset and a risky asset with stochastic returns $\mathbf{r} \in \Re_+^S$. If the price of the safe asset is normalized to one and the price of the risky asset is p, then, ruling out any short sales, her state-contingent returns are

$$Z(x) = \{\mathbf{z} : z_s = w - xp + xr_s, \quad s \in S\}.$$

The action variable $x \geq 0$ is a scalar reflecting the quantity of the risky asset that the individual purchases.[2]

Our second example is stochastic production characterized by a stochastic production function of a single variable input

$$z_s \leq f(x, \varepsilon_s)$$

[1]Hence, in this paper we do not analyze the case where the realization of the technology is random, even though actual production is.

[2]Note that this version of the portfolio problem does not satisfy the free disposability assumption, Z.2, imposed below.

where x is the scalar input and ε_s is a stochastic input beyond the producer's control. Thus,

$$Z(x) = \{\mathbf{z} : z_s \leq f(x, \varepsilon_s), \quad s \in S\}.$$

Here, for example, x might be interpreted as the level of investment that a firm undertakes prior to the resolution of uncertainty.

An alternative specification of a stochastic technology is offered by

$$Z(\mathbf{x}) = \left\{\mathbf{z} : \sum_s \gamma_s z_s \leq g(\mathbf{x})\right\},$$

where $\mathbf{x}$ is a vector of inputs and $\gamma_s > 0$ for all s. In what follows, we shall refer to this as the state-contingent transformation function case. Finally, one could also consider convolutions of the standard portfolio problem and either of the alternative representations of stochastic production such as that considered in the simultaneous choice of stochastic production levels and involvement in a futures or forward market (Danthine, 1978; Holthausen, 1979; Anderson and Danthine, 1983; Chambers and Quiggin, 2002).

We impose the following properties on the event-contingent output correspondence.

Properties of $Z(\mathbf{x})$ (Z)

Z.1 $\mathbf{0}_S \in Z(\mathbf{x})$ for all $\mathbf{x} \in \Re^M_+, \mathbf{z} \notin Z(\mathbf{0}_N)$ for $\mathbf{z} \neq \mathbf{0}_S$;

Z.2 $\mathbf{z}' \leq \ \mathbf{z} \in Z(\mathbf{x}) \Rightarrow \mathbf{z}' \in Z(\mathbf{x})$;

Z.3 $Z(\mathbf{x})$ is bounded for all $\mathbf{x} \in \Re^M_+$ (boundedness);

Z.4. $Z(\mathbf{x})$ is convex for all $\mathbf{x} \in \Re^M_+$;

Z.5. Z is a continuous correspondence.

The first part of Z.1 allows for the possibility of output inaction. Regardless of the level of inputs committed, the producer can always choose to produce nothing in each state of nature. The second part requires a positive commitment of inputs to realize a positive output. Z.3 and Z.5 are technical assumptions made to facilitate analysis, while Z.4 intuitively requires that state-contingent output sets exhibit an increasing 'marginal rate of transformation' between state-contingent outputs.

2.3 The utility correspondence and the utility set

Now suppose that the producer is concerned to maximize an objective function that is monotonically increasing in $\mathbf{u} \in \Re_{+}^{S}$ with typical element

$$u_s = u(z_s, \mathbf{x})$$

for some function $u : \Re_{+} \times \Re_{+}^{M} \rightarrow \Re_{+}$ that is continuous, concave, monotone increasing in z_s and normalized to satisfy $u(\mathbf{0}) = 0$. Define the *utility correspondence* by

$$U(\mathbf{x}) = \{\mathbf{u}(\mathbf{z}, \mathbf{x}) \in \Re_{+}^{S} : \quad \mathbf{z} \in Z(\mathbf{x})\}.$$

Lemma 1 *If Z satisfies Z.1-4, then U satisfies:*

U.1 $\mathbf{u}(0_S, \mathbf{x}) \in U(\mathbf{x})$ for all $\mathbf{x} \in \Re_{+}^{M}$;

U.2 $\mathbf{u}' \leq \ \mathbf{u} \in U(\mathbf{x}) \Rightarrow \mathbf{u}' \in U(\mathbf{x})$;

U.3 $U(\mathbf{x})$ is bounded for all $\mathbf{x} \in \Re_{+}^{M}$;

U.4 $U(\mathbf{x})$ is convex for all $\mathbf{x} \in \Re_{+}^{M}$;

U.5 U is a continuous correspondence.

Proof: *All proofs are in the appendix.*

Our example technologies illustrate. In the simple portfolio selection problem:

$$U(x) = \{\mathbf{u} : u_s = u(w - xp + xr_s), \quad s \in S\},$$

where u is a standard von Neumann–Morgenstern utility function. In the stochastic production function case for the Sandmovian model, assuming that the firm must incur a nonstochastic cost of v to engage effort x :

$$U(x) = \{\mathbf{u} : u_s \leq u(f(x, \varepsilon_s) - vx), \quad s \in S\},$$

where again u is a von-Neumann–Morgenstern utility function. Both the portfolio-selection model and the stochastic production function representation are thus special cases of a utility correspondence of the general form

$$U(x) = \{\mathbf{u} : u_s \leq u(h(x, \varepsilon_s)), \quad s \in S\}.$$

The Sandmovian model with a stochastic production function generalizes the simple portfolio selection model by allowing the action variable to affect the returns from the risky asset nonlinearly. This similarity allows results developed for the Sandmovian model with a stochastic production function to be extended directly to the simple portfolio selection problem (Athey, 2000, 2002). Conversely, Milgrom (1994) has shown that results developed in the simple portfolio selection problem can be extended directly to the Sandmovian problem under price uncertainty (this corresponds to the case where the stochastic component enters multiplicatively in the stochastic production function) under general conditions.

For the convolution of the Sandmovian model and the state-contingent transformation function, we have

$$U(\mathbf{x}) = \left\{ \mathbf{u} : u_s \le u(z_s - \mathbf{vx}), \quad s \in S, \ \sum_s \gamma_s z_s \le g(\mathbf{x}) \right\},$$

where $\mathbf{v}$ is a vector of nonstochastic input prices.

The *utility set,* $\mathcal{U} \subset \Re^S_+$, is the range of the utility correspondence and can be identified intuitively with the utility-possibilities frontier familiar from non-stochastic welfare analysis:

$$\mathcal{U} = \cup_{\mathbf{x}} U(\mathbf{x}).$$

Although each $U(\mathbf{x})$ is a closed, convex set, $\mathcal{U}$ generally is not. Denote the closure of the convex hull of $\mathcal{U}$ by

$$U^* = cl\,(conv\,(\mathcal{U})).$$

2.4 Some definitions

Our discussion frequently relies on some concepts about convex sets (Rockafellar, 1970).[3] For a convex set A, $F \subset A$ is an *exposed face* if there exists a supporting hyperplane to A, P, such that

$$P \cap A = F.$$

[3] We restrict attention to convex sets, $A \subseteq \Re^S$.

An exposed face that consists of a single point, $F = \{a\}$, is called an *exposed point.* $a \in A$ is an *exposed kink* if it belongs to more than one exposed face in A. For an exposed kink, the notation $a\{B\}$ means that a belongs to the exposed faces for each of the hyperplanes in the set B. Figures 1a, 1b, and 1c illustrate each of these concepts.

For any convex set, A, its *free disposal hull* $D(A)$ is the set of points dominated by the elements of A

$$D(A) = \{b : b \leq a, \quad a \in A\}.$$

C is said to be a *cube* if it is the free disposal hull of a single point a, that is,

$$C = D(\{a\}).$$

We shall refer to a as the *outer vertex* of C in this case.

Suppose two convex sets A and C have intersecting exposed faces F and F', respectively, for a common supporting hyperplane P. That is, there is a P that supports A and C and that satisfies

$$P \cap A = F,$$

$$P \cap C = F',$$

and

$$F \cap F' \neq \varnothing.$$

In this case, C is a *contraction of* A if $C \subset A$.

Now, consider these definitions in terms our example technologies. For the simple portfolio-selection problem $Z(x)$ consists of a single point in $\Re_+^S$. It is, therefore, an exposed face, an exposed point, and an exposed kink. $Z(x)$ for the stochastic production function case corresponds to the free disposal hull of the single point in $\Re_+^S$ $(z_s = f(x, \varepsilon_s), s \in \Omega)$. $Z(x)$ is thus a cube, and its outer vertex, $(z_s = f(x, \varepsilon_s), s \in \Omega)$, is simultaneously an exposed face, an exposed point, and an exposed kink. It is also straightforward to show that $U(x)$ for each of these specifications inherits these characteristics from its parent

$Z(x)$. The cubical form of $Z(x)$ for these specifications reflects the fact that the technology for both of these specifications does not permit substitutability between state-contingent outputs once a fixed level of effort is committed (Chambers and Quiggin, 2000). For the simple portfolio-selection model, this is a natural consequence of the fact that the individual is a small participant in a securities market, who necessarily takes the asset structure as given. However, in the case of the stochastic production function, it represents an arbitrary restriction on the stochastic technology which excludes the substitutability of state-contingent outputs from consideratoin.

To see that not all state-contingent technologies lead to cubical utility correspondences, notice that $Z(\mathbf{x})$ for the state-contingent transformation function is the free disposal hull of a single hyperplane in $\Re_+^S$ (a half space). All points on the hyperplane, that is,

$$\left\{\mathbf{z} : \sum_s \gamma_s z_s = g(\mathbf{x})\right\}$$

belong to the same exposed face, and the points where this hyperplane intersect the axes correspond to exposed kinks. One can easily generalize this technology by replacing the linear combination of state-contingent outputs by a suitably smooth convex function of the outputs. Doing so would then yield a technology which possesses exposed points which are not also exposed kinks. Moreover, even in the absence of this generalization, the $U(\mathbf{x})$ that emerges from applying the Sandmovian net-returns specification to this technology does not inherit this linearity property. For u strictly concave, its outer boundary will consist of a smooth curve that is concave to the origin. Points away from the axes on this curve will represent exposed points that are neither exposed faces nor exposed kinks. This smoothness of the frontier reflects the individual's ability to substitute between state-contingent outputs for a fixed level of inputs.

To illustrate further the relationship between stochastic technologies, for the scalar input case consider a stochastic production function $f(x, \varepsilon_s)$ and a state-contingent transformation function defined so that for given x,

$$\sum_s \gamma_s z_s \leq \sum_s \gamma_s f(x, \varepsilon_s) = g(x).$$

$Z(x)$ for the stochastic production function, which generates this state-contingent transformation function, is itself a cube. It also is a contraction of the output set for the generated state-contingent transformation function. More generally, consider any convex technology, $\hat{Z}(x)$, having $\mathbf{z} = [f(x, \varepsilon_1), ..., f(x, \varepsilon_S)]$ as an exposed point with supporting hyperplane defined by the vector γ. This technology is always a contraction of the output set for the associated state-contingent transformation function, while the output set for the stochastic production function always turns out to be a contraction of $\hat{Z}(x)$.

3 Signals and probabilities

The set-up here closely parallels that used in standard discrete-state-space representations of the information problem (Hirshleifer and Riley, 1992). Because the state space is of the form $\Omega = S \times N$, the probability distribution may be represented by an $S \times N$ matrix P with entry p_{sn} corresponding to the joint probability of production event s and signal n. Let

$$\pi_s^0 = \sum_n p_{sn}$$

be the unconditional probability of production event s,

$$\pi_n = \sum_s p_{sn}$$

be the unconditional probability of signal realization n,[4] and

$$\pi_s^n = \frac{p_{sn}}{\pi_n}$$

be the probability of event s conditional on the observation of signal realization n. Then $\boldsymbol{\pi}^0$ denotes the vector of unconditional probabilities or *prior* probabilities and $\boldsymbol{\pi}^n$ denotes the vector of probabilities conditional on the observation of signal realization n, or the *posterior* probabilities. We confine attention to the strictly positive elements of the unit simplex, which we denote by $\mathcal{P} \subset \Re_{++}^S$. Thus, the signal is not perfectly informative regarding any event, and for all s and n, $0 < \pi_s^n < 1$. We will refer to the set of joint, prior, and posterior probabilities as an *information structure* and denote it by θ . We denote the set of prior and posterior probabilities for the information structure θ as $\Pi(\theta) = \left\{\boldsymbol{\pi}^0, \boldsymbol{\pi}^1, ..., \boldsymbol{\pi}^N\right\}$.

[4] More properly, we should refer to this as the signal realization from the finest partition.

4 The expected-utility function

We assume that the producer's objective function is of the expected utility or 'linear-in-probabilities' form

$$\sum_{s=1}^{S} \pi_s u(z_s, \mathbf{x}).$$

For a given preference ordering over $\Re_+^S \times \Re_+^M$ of feasible outcomes $(\mathbf{z}, \mathbf{x})$, consistent with the expected-utility representation, the choice of u is determined up to an affine transformation. It is therefore possible to choose an appropriate normalization for the purpose of analytical tractability. Given an initial position consisting of a probability vector $\boldsymbol{\pi}^0$ and an input-output vector $(\mathbf{z}^0, \mathbf{x}^0)$, a natural normalization is $u(0, \mathbf{0}) = 0$, $\sum_s \pi_s^0 u(z_s^0, \mathbf{x}^0) = 1$.

Define the *input-restricted expected utility function,*[5] $V : \mathcal{P} \times \Re_+^M \to \Re_+$, by

$$V(\boldsymbol{\pi}, \mathbf{x}) = Max_{\mathbf{u}} \left\{ \sum_{s=1}^{S} \pi_s u_s : \mathbf{u} \in U(\mathbf{x}) \right\}.$$

The input-restricted expected utility function is the (maximal) support function for $U(\mathbf{x})$. Consequently, it is entirely analogous to a 'revenue function' for $U(\mathbf{x})$ given the 'prices', $\boldsymbol{\pi}$.

It is particularly interesting to note that in the case of either the simple portfolio selection model or the stochastic production function representation, the input-restricted expected utility function corresponds to a trivial maximization problem, whence

$$V(\boldsymbol{\pi}, x) = \sum_{s=1}^{S} \pi_s u(h(x, \varepsilon_s)).$$

When the technology is given by the state-contingent production transformation specification and other more general specifications, isolating the input-restricted expected utility function involves a less trivial maximization problem. This, perhaps, can be seen most intuitively by considering the convolution of the state-contingent product transformation specification and a separable effort utility specification, that is,

$$u(z_s - \mathbf{v}\mathbf{x}) = \hat{u}(z_s) - v(\mathbf{x}).$$

[5] We thank David Hennessy for suggesting this terminology.

In this case, we have

$$V(\boldsymbol{\pi},\mathbf{x}) = N(\boldsymbol{\gamma}, g(\mathbf{x}); \boldsymbol{\pi}) - v(\mathbf{x}),$$

where $N(\boldsymbol{\pi}, g(\mathbf{x}))$ is the indirect utility function familiar from nonstochastic consumer theory when preferences are additively separable:

$$N(\boldsymbol{\gamma}, g(\mathbf{x}); \boldsymbol{\pi}) = \sup_{\mathbf{z}} \left\{ \sum_{s=1}^{S} \pi_s \hat{u}(z_s) : \sum_s \gamma_s z_s \leq g(\mathbf{x}) \right\}.$$

N is convex in $\boldsymbol{\pi}$, quasi-convex in $\boldsymbol{\gamma}$, and, given strict risk aversion, concave in g.

The *expected utility function,* $W : \mathcal{P} \to \Re_+$ is defined by

$$\begin{aligned} W(\boldsymbol{\pi}) &= \sup \left\{ \sum_{s=1}^{S} \pi_s u_s : \mathbf{u} \in \mathcal{U} \right\} \\ &= \sup_{\mathbf{x}} \{ V(\boldsymbol{\pi},\mathbf{x}) \}. \end{aligned}$$

We assume that solutions always exist and that $V(\boldsymbol{\pi},\mathbf{x})$ and $W(\boldsymbol{\pi})$ are finite.

Both $V(\boldsymbol{\pi},\mathbf{x})$ and $W(\boldsymbol{\pi})$ are convex and continuous on $\mathcal{P}$ (Färe, 1988). Denote

$$\bar{\mathbf{u}}(\boldsymbol{\pi},\mathbf{x}) = \arg\max \left\{ \sum_{s=1}^{S} \pi_s u_s : \mathbf{u} \in U(\mathbf{x}) \right\},$$

$$\mathbf{u}(\boldsymbol{\pi}) = \arg\sup \left\{ \sum_{s=1}^{S} \pi_s u_s : \mathbf{u} \in \mathcal{U} \right\},$$

and

$$\mathbf{x}(\boldsymbol{\pi}) \in \arg\sup \{ V(\boldsymbol{\pi},\mathbf{x}) \}.$$

$\bar{\mathbf{u}}(\boldsymbol{\pi},\mathbf{x})$ and $\mathbf{u}(\boldsymbol{\pi})$ are exposed faces of $U(\mathbf{x})$ and U^*, respectively, defined by hyperplanes with normal $\boldsymbol{\pi}$. By analogy with the theory of profit maximization, in the case when they are singletons, one may think of $\bar{\mathbf{u}}(\boldsymbol{\pi},\mathbf{x})$ and $\mathbf{u}(\boldsymbol{\pi})$ as vectors of supplies of state-contingent utilities.

Because $U(\mathbf{x})$ is convex and satisfies U.1–U.5, it may be recaptured from V by applying the duality mapping (Färe, 1988)

$$U(\mathbf{x}) = \cap_{\pi} \left\{ \mathbf{u} : \sum_{s=1}^{S} \pi_s u_s \leq V(\boldsymbol{\pi},\mathbf{x}) \right\}.$$

Generally, it will not be possible to recapture $\mathcal{U}$ by a similar duality mapping. However, W is the support function for U^*, whence

$$U^* = \cap_{\boldsymbol{\pi}} \left\{ \mathbf{u} : \sum_{s=1}^{S} \pi_s u_s \leq W(\boldsymbol{\pi}) \right\}.$$

U^* is observationally equivalent to $\mathcal{U}$ for individuals with linear-in-probability preferences.

5 The value of information

As usually defined, the value of information for a productive firm is the difference between its optimal expected utility as evaluated using the prior probabilities and the *ex ante* value of expected utility obtained from the posterior distribution. We adopt the same definition here and distinguish two notions of the value of information: the *value of the nth signal* and *the value of the information structure* θ . We will also distinguish between the case where the input bundle is fixed and the case where the input bundle is freely variable in response to the reception of different signals.

The latter decomposition is important for several reasons. First, it permits a convenient analytic decomposition that is particularly appropriate for state-contingent models. One can visualize the firm, in much the same fashion as the Slutsky decomposition of a consumer's response to a price change, as responding to the provision of information by first moving around its transformation curve to get the most out of its current input bundle, and then adjusting that input bundle optimally. The analogy with the Slutsky decomposition becomes more apposite when one recalls that probabilities are interpretable as the producer's internal prices of state-contingent utilities.

Even more importantly, however, such measures are of practical concern. For example, consider the case of a firm that is contractually obligated, either as the result of negotiations with a union or other firms, to employ a fixed bundle of variable inputs in combination with its quasi-fixed capital stock. Is information ever of value to the producer after he has made these input commitments? Intuitively, one would think that the answer would be yes, but less valuable then when the producer has free rein to adjust his inputs optimally. This decomposition permits a formal analysis of such effects, and as we show below, the most familiar specification for risk-averse firms forces the answer to our rhetorical question to be no.

The *value of the nth signal* for the fixed input bundle, $\mathbf{x}$, is

$$\bar{\triangle}\left(\boldsymbol{\pi}^{n},\boldsymbol{\pi}^{0},\mathbf{x}\right)=V\left(\boldsymbol{\pi}^{n},\mathbf{x}\right)-\sup_{\mathbf{u}\in\bar{\mathbf{u}}(\boldsymbol{\pi}^{0},\mathbf{x})}\left\{\sum_{s}\pi_{s}^{n}u_{s}\right\}\geq 0,$$

where the inequality follows by the definition of V. In words, $\bar{\triangle}\left(\boldsymbol{\pi}^{n},\boldsymbol{\pi}^{0},\mathbf{x}\right)$ is the difference between the optimal expected utility, as calculated for the posterior probabilities, and the producer's best alternative, using the posterior probabilities, given the input commitment he has made for the prior probabilities.

The *value of the information structure* θ *for the fixed input bundle* $\mathbf{x}$, $\bar{I}\left(\theta,\mathbf{x}\right)$, is the difference between the producer's *ex ante* expected utility when he has access to the entire information structure and his expected utility when he knows only the prior probabilities. More formally,

$$\begin{aligned}\bar{I}\left(\theta,\mathbf{x}\right) &= \sum_{n}\pi_{n}\bar{\triangle}\left(\boldsymbol{\pi}^{n},\boldsymbol{\pi}^{0},\mathbf{x}\right)\\ &= \sum_{n}\pi_{n}V\left(\boldsymbol{\pi}^{n},\mathbf{x}\right)-V\left(\boldsymbol{\pi}^{0},\mathbf{x}\right)\geq 0,\end{aligned}$$

where second equality follows because $\boldsymbol{\pi}^{0}=\sum_{n}\pi_{n}\boldsymbol{\pi}^{n}$.

Similarly, the *value of the nth signal* is defined

$$\triangle\left(\boldsymbol{\pi}^{n},\boldsymbol{\pi}^{0}\right)=W\left(\boldsymbol{\pi}^{n}\right)-\sup_{\mathbf{u}\in\mathbf{u}(\boldsymbol{\pi}^{0})}\left\{\sum_{s}\pi_{s}^{n}u_{s}\right\},$$

and the value of the information structure θ is

$$\begin{aligned}I\left(\theta\right) &= \sum_{n}\pi_{n}\triangle\left(\boldsymbol{\pi}^{n},\boldsymbol{\pi}^{0}\right)\\ &= \sum_{n}\pi_{n}W\left(\boldsymbol{\pi}^{n}\right)-W\left(\boldsymbol{\pi}^{0}\right)\geq 0.\end{aligned}$$

6 Bounds on the value of information

We now consider bounds on the value of signals and the value of the information structure for a given technology and preferences. The classic results of Blackwell (1951) show that, for expected-utility preferences, the value of information can never be negative. This result follows directly from the fact that expected-utility preferences are linear in the probability vector π. The Blackwell result relies on the assumption of expected-utility maximization which implies that preferences are linear in probabilities.

The fact that zero is a lower bound for the value of information is well-known. By contrast, to our knowledge, there has been relatively little discussion of upper bounds, or of the conditions under which upper and lower bounds are attained. Theorem 2 addresses the first issue.

Theorem 2

$$V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\left(\max\left\{\frac{\pi_1^n}{\pi_1^0}, ..., \frac{\pi_S^n}{\pi_S^0}\right\} - 1\right) \geq \bar{\triangle}\left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0, \mathbf{x}\right) \geq 0,$$

$$\sum_n \boldsymbol{\pi}_n V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\left(\max\left\{\frac{p_{1n}}{\pi_1^0}, ..., \frac{p_{Sn}}{\pi_S^0}\right\} - 1\right) \geq \bar{I}\left(\theta, \mathbf{x}\right) \geq 0,$$

$$W\left(\boldsymbol{\pi}^0\right)\left(\max\left\{\frac{\pi_1^n}{\pi_1^0}, ..., \frac{\pi_S^n}{\pi_S^0}\right\} - 1\right) \geq \triangle\left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0\right) \geq 0,$$

$$\sum_n \boldsymbol{\pi}_n W\left(\boldsymbol{\pi}^0\right)\left(\max\left\{\frac{p_{n1}}{\pi_1^0}, ..., \frac{p_{nS}}{\pi_S^0}\right\} - 1\right) \geq I\left(\theta\right) \geq 0.$$

Signals and information generally are most valuable when the input-restricted expected utility function and the expected utility function take, respectively, the forms $V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\max\left\{\frac{\pi_1}{\pi_1^0}, ..., \frac{\pi_S}{\pi_S^0}\right\}$ and $W\left(\boldsymbol{\pi}^0\right)\max\left\{\frac{\pi_1}{\pi_1^0}, ..., \frac{\pi_S}{\pi_S^0}\right\}$. This corresponds naturally to the case where the frontiers of U and U^* are linear. In that case, even the smallest changes in probability distributions will lead to large changes in the choice of state-contingent utilities and hence in state-contingent outputs.

On the other hand, signals and information are least valuable when the expected utility function is linear over a range of probability distributions. By standard results from duality theory (McFadden, 1978), linearity in the expected utility function maps directly into kinks in the frontier of U or U^*. The existence of a kink implies that the producer takes the same action in response to a range of possible signals. The linearity property of expected utility then implies that the lower bound of zero is attained.

We pursue this point formally in the following theorem and corollaries.

Theorem 3 *$\bar{\triangle}(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0, \mathbf{x}) = 0$ if and only if $\bar{\mathbf{u}}(\boldsymbol{\pi}^0, \mathbf{x})$ contains an exposed kink in $U(\mathbf{x})$ for $(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0)$. $\bar{I}(\theta, \mathbf{x}) = 0$ if and only if $U(\mathbf{x})$ is a contraction of*

$$\cap_{\pi\in\Pi(\theta)}\left\{\mathbf{u}:\sum_s \pi_s u_s \leq \sup_{\bar{\mathbf{u}}(\pi^0,\mathbf{x})}\left\{\sum_s \pi_s u_s\right\}\right\}.$$

Corollary 4 *Suppose $\bar{\mathbf{u}}(\boldsymbol{\pi}^0, \mathbf{x})$ is unique, then $\bar{I}(\theta, \mathbf{x}) = 0$ if and only if $\bar{\mathbf{u}}(\boldsymbol{\pi}^0, \mathbf{x})$ is an exposed kink for $\Pi(\theta)$.*

Corollary 5 *$\bar{I}(\theta, \mathbf{x}) = 0$ for all information structures θ if and only if $U(\mathbf{x})$ is a cube.*

It is well-known that the value of information is always nonnegative. In addition, we derive an important consequence of the preceding theorem and corollaries for the most commonly studied class of models, that is:

$$V(\boldsymbol{\pi}, x) = \sum_{s=1}^{S} \pi_s u\left(h\left(x, \varepsilon_s\right)\right).$$

We have:

Corollary 6 *If $V(\boldsymbol{\pi}, x) = \sum_{s=1}^{S} \pi_s u\left(h\left(x, \varepsilon_s\right)\right)$, the lower bounds on $\bar{\triangle}(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0, \mathbf{x})$ and $\bar{I}(\theta, \mathbf{x})$ are achieved for all information structures.*

In either the portfolio selection model or the stochastic production function representation, acquiring information is worthless unless the producer can simultaneously adjust his actions. Thus, this class of models allows information to have only a minimal impact on the risk-averse firm. In the portfolio selection model, this is a natural consequence of the fact that a small individual cannot, by definition, affect the risky assets that are available in the market. The discipline of the market limits his or her range of actions, and his or her only possible response to the acquisition of new information is to alter the portfolio mix. Without that ability, the new information is valueless.

In the stochastic production specification, this inability to adjust emerges not from market forces, but from an arbitrary restriction on the technology. Producers in this model don't respond to the new information not because they don't want to, but because they have been assumed not to. If this structural restriction were accurate, it would imply, for example, that firms which are contractually or otherwise obligated to a particular input bundle would not value, and therefore not expend positive resources to acquire finer information. This seems implausible and suggests that models of this sort should be viewed as a polar case, which minimizes the role that information can play. Certainly models employing this assumption will systematically underestimate the true economic value of information. Thus, empirical models of the value of information based upon such a specification have an identifiable downward bias in their evaluations.

Corollary 6 also explains the close connection that has emerged between results on the value of information, the response of input utilization to new information, and various forms of supermodularity of the objective function in the generalized portfolio model.[6] In this specification, *nothing* can happen unless the input adjusts, otherwise the information is always valueless. Moreover, once the input is determined, the complete rank-ordering of state-contingent outputs and utilities is known, and given appropriate supermodularity or log supermodularity restrictions, so is the rank-ordering of the state-contingent marginal utilities of input variation. Hence, any tightening of the output distribution or the $\mathbf{u}$ distribution is inextricably entangled with an expansion effect on outputs or utilities caused by the input adjustment. Thus, determining the impact of new information, in the form of a signal, on input utilization or the level of investment then reduces to determining how that new signal changes the expected value of the rank-ordered (by supermodularity) state-contingent marginal utilities.

To appreciate how the ability to substitute between state-contingent output interacts with the producer's attitudes towards risk in determining the value of information in a more general model, let us consider, for concreteness, the separable utility version of the state-contingent transformation function under polar assumptions about risk aversion. For the limiting case where $\hat{u}(z) = z$ (risk neutrality), we obtain

$$V(\boldsymbol{\pi}, \mathbf{x}) = g(\mathbf{x}) \, Max \left\{ \frac{\pi_1}{\gamma_1}, \ldots, \frac{\pi_S}{\gamma_S} \right\} - v(\mathbf{x}),$$

[6]These findings are exhaustively categorized in a series of elegant papers by Athey (2002) and Athey and Levin (1998, 1999). Although their specification is for a continuous state space and is in terms of a parametrized distribution function, in a finite state space, it can be made formally equivalent to ours by inducing distributions for the state contingent outputs from the technology and the probabilities.

so that

$$\bar{I}(\theta,\mathbf{x}) = g(\mathbf{x})\left[\sum_n \pi_n Max\left\{\frac{\pi_1^n}{\gamma_1},...,\frac{\pi_S^n}{\gamma_S}\right\} - Max\left\{\frac{\pi_1^0}{\gamma_1},...,\frac{\pi_S^0}{\gamma_S}\right\}\right],$$

which is positive so long as $g(\mathbf{x})$ is. For the polar case of complete aversion to risk, we obtain, in the limit,

$$V(\boldsymbol{\pi},\mathbf{x}) = \frac{g(\mathbf{x})}{\sum_s \gamma_s} - v(\mathbf{x}),$$

and $\bar{I}(\theta,\mathbf{x}) = 0$ for all $\mathbf{x}$ (including the optimal $\mathbf{x}$). In the latter case, information is not valuable to the producer because it never affects his desired consumption pattern, which is chosen to eliminate variation in consumption. Hence, as in Corollary 6, we always achieve the lower bound on the value of information.

Turning to the overall value of information, exactly parallel arguments reveal:

Theorem 7 $\triangle(\boldsymbol{\pi}^n,\boldsymbol{\pi}^0) = 0$ *if and only if* $\mathbf{u}(\boldsymbol{\pi}^0)$ *contains an exposed kink in* U^* *for* $(\boldsymbol{\pi}^n,\boldsymbol{\pi}^0)$. $I(\theta) = 0$ *if and only if* U^* *is a contraction of*

$$\cap_{\boldsymbol{\pi}\in\Pi(\theta)}\left\{\mathbf{u} : \sum_s \pi_s u_s \leq \sup_{\mathbf{u}(\boldsymbol{\pi}^0)}\left\{\sum_s \pi_s u_s\right\}\right\}.$$

Corollary 8 *Suppose* W *is differentiable at* $\boldsymbol{\pi}^0$, *then* $I(\theta) = 0$ *if and only if* $\mathbf{u}(\boldsymbol{\pi}^0)$ *is an exposed kink for* $\Pi(\theta)$.

Corollary 9 $I(\theta) = 0$ *for all possible information structures if and only if* U^* *is a cube.*

When W is not differentiable at $\boldsymbol{\pi}^0$, there is not an unique optimal solution for the prior probabilities. Hence, in that case, the exposed face of U^* must be at least locally linear. If either a signal or an information structure is to be valueless, however, the hyperplanes defined by the associated posterior distributions must pass through at least one exposed kink in the exposed face. When this happens, varying the posterior probabilities brings no adjustment on the part of the producer. However, when W is differentiable at $\boldsymbol{\pi}^0$, information can only be valueless if the optimal solution for the prior distribution corresponds to an exposed kink for the entire range of posterior probabilities.

We conclude this discussion on the general value of information with

Corollary 10 *For an information structure θ, the following are equivalent:*

(i) $I(\theta) = 0$;

(ii) W is linear on $\Pi(\theta)$;

(iii) For all $\boldsymbol{\pi}^n \in \Pi(\theta)$, $\Delta(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0) = 0$.

Conversely, the upper bound in Theorem 2 is attained when

$$U^* = \left\{ \mathbf{u} : \sum_s \pi_s^0 u_s \leq W\left(\boldsymbol{\pi}^0\right) \right\}.$$

These results illustrate the duality between the utility correspondence U^* and the behavior of the preference function W on $\Pi(\theta)$. When W is linear, U^* is cubical and information is valueless. Conversely, information attains its maximal value when the boundary of U^* is a hyperplane passing through $W\left(\boldsymbol{\pi}^0\right)$. Our final result identifies a general condition on the stochastic production function specification to yield zero information generally, that is, for U^* to be cubical.

Theorem 11 *For the specification*

$$V(\boldsymbol{\pi}, x) = \sum_s \pi_s u\left(h\left(x, \varepsilon_s\right)\right),$$

information always has zero value if

$$\begin{aligned} W(\boldsymbol{\pi}) &= \sup\left\{\sum_s \pi_s u\left(h\left(x, \varepsilon_s\right)\right)\right\} \\ &= \sum_s \pi_s \sup_x \left\{u\left(h\left(x, \varepsilon_s\right)\right)\right\}. \end{aligned}$$

Corollary 12 *If $u\left(h\left(x, \varepsilon_s\right)\right) = f\left(m(x) + g\left(\varepsilon_s\right)\right)$ with f monotonic increasing, then information is always valueless.*

The corollary shows that information is always valueless, for example, for the case of a stochastic production function with additive error. Also notice that the corollary and the Theorem remain true if scalar x is replaced by a vector $\mathbf{x}$. The upper bounds derived depend on the assumption that the outcome space is assumed to be $\Re_+$. Broadly speaking, this is equivalent to ruling out short-selling in a portfolio choice problem. In general, if the outcome space is unbounded, and the utility correspondence is linear (that is, bounded by a hyperplane), the value of information is unbounded.

7 Concluding comments

Much previous analysis of the value of information for a risk-averse producing firm has focused on cases where the recipients of information must select a single scalar decision variable, which may be either an element of an interval in the real line or a member of a discrete set of options. In particular, in the case of production decisions, the assumed technology has almost universally been a stochastic production function, in which the set of feasible state-contingent vectors is a one-dimensional manifold, indexed by the input level.

In such a setting, many of the most important uses of information cannot be represented directly. For example, the observation of a signal that reduces the posterior probability of one event and increases that of another does not give rise to opportunities to 'bet' on the event with increased likelihood, by increasing output conditional on that event while reducing output conditional on the event with reduced likelihood. The only feasible response is to increase effort and output in response to 'good' news, while reducing effort and output in response to 'bad' news. Limits on the value of information therefore arise, in large measure, from limits on the range of actions available to producers.

As in many other problems, a more general representation is also more tractable. The stochastic production function technology is a special case of the general state-contingent production technology first modelled by Arrow and Debreu. For this general model, it is possible to exploit the idea that relative probabilities behave like relative prices, with the associated duality relationships. Observation of a signal is similar to a change in relative prices, which in the state-contingent production framework yields a natural characterization of the value of information in terms of convex sets. In particular, it is straightforward to derive upper and lower bounds for the value of information and rankings of the valuation for alternative problem settings and information structures.

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9 Appendix: Proofs of Results

Proof of Lemma 1: U.1 follows from Z.1. U.2 is a consequence of the monotonicity of the utility function and Z.2. U.3 follows by the continuity of the utility function and Z.3. Suppose that $\mathbf{z}^0, \mathbf{z}' \in Z(\mathbf{x})$, then, because u is concave,

$$\lambda u\left(z_s^0, \mathbf{x}\right) + (1-\lambda)\, u\left(z_s'\right) \leq u\left(\lambda z_s^0 + (1-\lambda)\, z_s', \mathbf{x}\right),$$

for all s.The right-hand side belongs to $U(\mathbf{x})$ by Z.4, and the left-hand side, along with U.2. establishes U.4. U.5 follows from the continuity of Z and the continuity of u .

Proof of Theorem 2: The proof is for $\bar{\triangle}\left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0, \mathbf{x}\right)$. An exactly parallel proof applies to the remaining expressions. By the definition of V,

$$U(\mathbf{x}) \subset \left\{\mathbf{u} : \sum_s \pi_s^0 u_s \leq V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\right\}.$$

Thus, $V(\boldsymbol{\pi}, \mathbf{x})$ must be bounded from above by

$$\begin{aligned}\bar{V}(\boldsymbol{\pi}, \mathbf{x}) &= \max\left\{\sum_s \pi_s u_s \colon \sum_s \pi_s^0 u_s \leq V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\right\}\\ &= V\left(\boldsymbol{\pi}^0, \mathbf{x}\right) \max\left\{\frac{\pi_1}{\pi_1^0}, ..., \frac{\pi_S}{\pi_S^0}\right\},\end{aligned}$$

and hence the maximum value of information is

$$\bar{\triangle}\left(\boldsymbol{\pi}, \boldsymbol{\pi}^0, \mathbf{x}\right) = \bar{V}(\boldsymbol{\pi}, \mathbf{x}) - V\left(\boldsymbol{\pi}^0, \mathbf{x}\right) = V\left(\boldsymbol{\pi}^0, \mathbf{x}\right)\left[\max\left\{\frac{\pi_1}{\pi_1^0}, ..., \frac{\pi_S}{\pi_S^0}\right\} - 1\right].$$

Proof of Theorem 3: Suppose

$$V\left(\boldsymbol{\pi}^n, \mathbf{x}\right) - \sup_{\bar{\mathbf{u}}(\boldsymbol{\pi}^0, \mathbf{x})}\left\{\sum_s \pi_s^n u_s\right\} = 0.$$

Hence, $\bar{\mathbf{u}}\left(\boldsymbol{\pi}^0, \mathbf{x}\right)$ must contain at least one element that belongs to faces of $U(\mathbf{x})$ for both $\boldsymbol{\pi}^0$ and $\boldsymbol{\pi}^n$. Conversely, if $\bar{\mathbf{u}}\left(\boldsymbol{\pi}^0, \mathbf{x}\right)$ contains an exposed kink in $U(\mathbf{x})$ for $\left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0\right)$, $\bar{\triangle}\left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^0, \mathbf{x}\right)$ cannot be strictly positive. For the second part, suppose to the contrary that there exists a feasible $\mathbf{u}$ such that

$$\sum_s \pi_s^n u_s > \sup_{\bar{\mathbf{u}}(\boldsymbol{\pi}^0, \mathbf{x})}\left\{\sum_s \pi_s^n u_s\right\}$$

for any n. Then $\bar{I}(\theta, \mathbf{x}) > 0$.

Proof of Theorem 11: By Corollary 9, $I(\theta) = 0$ for all θ if and only if U^* is a cube. In this specification, for this to be true, there must then exist a single point $\tilde{\mathbf{u}} \in \Re^S$ (optimal for all $\boldsymbol{\pi}$) such that

$$u(h(x, \varepsilon_s)) \leq \tilde{u}_s$$

all x and s. This is always true if

$$u(h(x, \varepsilon_s)) \leq \sup_x \{u(h(x, \varepsilon_s))\} \leq \tilde{u}_s$$

all s.