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Asset pricing is a central topic of financial economics. There are two dominant approaches. *No-arbitrage pricing* starts from the assumption of a set of basis assets and imposes the absence of an arbitrage for those basis assets to deduce a set of state-claim prices (the *no-arbitrage prices*) that accurately and uniquely price assets lying in the linear subspace (the *market span*) generated by the basis assets. The no-arbitrage prices assign non-negative values to payouts in each state of Nature. Assets not lying in the linear subspace defined by the basis assets, however, are not uniquely priced. For such assets, the no-arbitrage prices define a range of potential prices. This range of asset prices is given by the interval between the *no-arbitrage bounds. Model-based pricing* (Bernardo and Ledoit, 2000) uses assumptions on an investor's preferences or production technology to deduce a singleton set of state-claim prices (in random variable terminology, a *stochastic discount factor*), which uniquely prices assets both in and out of the market span.

The strengths and weaknesses of the approaches reflect the assumptions used to generate them. The great strength of model-based pricing is that a correctly-specified model can uniquely determine the price for any asset characterized by a vector of state-contingent payoffs. However, the derivation of this asset price requires specific assumptions on preferences or technology (or both). Thus, the model-based approach lacks robustness. Moreover, model-based pricing has not, in general, proven successful in explaining observed asset prices. Hansen and Jagannathan (1997), for example, report disappointing performance results for a range of stochastic discount factor proxies.

No-arbitrage pricing, relying only on minimal assumptions, is quite general and is typically consistent with any meaningful equilibrium deduced from restrictions on consumer or producer behavior. But such generality has its cost, which is the inability of no-arbitrage pricing to determine unique prices for assets that are not perfectly replicable in the market. Moreover, the range of possible prices for such assets, determined by the eponymous *no-arbitrage bounds,* is frequently too broad to be useful.

The broadness of the no-arbitrage bounds has led to a number of suggestions on how to narrow them. Ross (1976), very early on, suggested bounding pricing residuals from the closely related arbitrage-pricing model by restricting portfolios to have no more than twice the market Sharpe ratio. Ledoit (1995) derived the pricing implications of excluding portfolios having Sharpe ratios higher than a prespecified level in arbitrage pricing theory. Cochrane and Saá-Requejo (2000) derive 'good-deal bounds' by using the Hansen-Jagannathan (1991) bounds on stochastic discount factors to generate variance (or second-moment) bounds on stochastic discount factors that are admissible in the construction of no-arbitrage asset prices. Bernardo and Ledoit (2000) similarly use a dual representation of bounds on the 'gain-loss ratio' to narrow the no-arbitrage bounds. These recommendations for narrowing the no-arbitrage bounds introduce reasonable restrictions on preferences, developed from general model-based approaches, into the calculation of the no-arbitrage bounds. Hence, they are *mixed approaches*, which combine elements of both no-arbitrage and model-based pricing.

This paper points out that another, apparently unexploited, opportunity exists for narrowing the no-arbitrage bounds. This opportunity does not require any assumption on individual preferences, and thus is logically independent of the consumption-based approach and the mixed approaches of Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000). More to the point, once these tighter bounds have been developed, consumption-based approaches can be applied to tighten them even further.

The no-arbitrage bounds, as typically calculated, are derived from models that ignore the real or productive component of the economy. Given the original thrust of this literature, this is quite understandable. But because many financial markets originally arose in response to vagaries associated with stochastic production processes, it is unrealistic. *Moreover, even if one has no interest in the real economy, ignoring it has unfortunate consequences for asset pricing* because there likely exists the potential for arbitrages between real and financial markets. Ruling out these arbitrages, of necessity, removes some of the ambiguity associated with the no-arbitrage bounds. Thus, if this information on the real economy can be incorporated in a reasonably tractable and informative fashion into asset pricing models, it should be. The purpose of this paper is to show how this can be done.

In what follows, we first present an intuitive overview of our main idea via a simple example. We then introduce some basic concepts and notation from convex analysis. The stochastic technology and financial market structures are then specified. Dual representations of the stochastic technology and of the derivation of no-arbitrage prices are then presented. We then formally address the issue of using information on the technology to narrow the no-arbitrage bounds. The key analytic concept is that of the derivative-cost function, which is used to define a notion of arbitrage that encompasses both the basis assets and stochastic production opportunities. No arbitrage prices, derived from this new notion of the absence of arbitrage, are shown to correspond to an appropriate subdifferential of the derivative-cost function, and the tighter no-arbitrage bounds to its directional (Gateaux) derivatives in the neigborhood of the origin. We then briefly compare our approach to other contributions, and the paper then concludes.

1 The Intuition

The kernel of our approach can be communicated with a simple example. Suppose that there exists a single asset in a two-period, two-state world. The period 0 price of the asset is 1, and the period 1 (stochastic) payouts on the asset equal a_i in each of the two states of Nature. Assume $a_2 > a_1 > 0$. The no-arbitrage present-value (state-claim) prices are the positive solutions, (q_1, q_2), to

$$q_1 a_1 + q_2 a_2 = 1.$$

Now consider pricing a call option, offered in period 0, on the asset with a strike price of K, where $a_2 > K > a_1$. The payout on the option is $(0, a_2 - K)$. Thus, the upper arbitrage bound on the option price is

$$\sup_{\mathbf{q}>\mathbf{0}} \{q_2 (a_2 - K) : q_1 a_1 + q_2 a_2 = 1\} = \frac{(a_2 - K)}{a_2} = 1 - \frac{K}{a_2} > 0,$$

while the lower arbitrage bound on the option price is, of course, 0.

Suppose, however, that there also exists a physical production technology, which transforms units of inputs committed in period 0 into random output (denominated in the same units as the payout for the asset) in period 1. Let the period 0 minimal cost function associated with the technology be given by

$$c(\mathbf{z}) = b_1 z_1 + b_2 z_2,$$

where z_i is the output in state of Nature i, and $b_i > 0$. Assume that $b_2 < \frac{1}{a_2}$ and $b_1 < \frac{1}{a_1}$.

If the option were to sell at its upper arbitrage bound of $1-\frac{K}{a_2}$, anyone having access to the technology could 'manufacture', or 'replicate by production', the option at a cost of $b_2(a_2-K)$ and sell it for a profit of $(a_2-K)\left(\frac{1}{a_2}-b_2\right)>0$. Therefore, the upper arbitrage bound could never prevail. Hence, $b_2(a_2-K)$ is a tighter upper bound on the option price than the original no-arbitrage bound.

Suppose, on the other hand, that the option were offered for sale at a price of 0. Individuals can now purchase at zero cost an asset returning $(0, a_2-K)$. If an individual with access to the technology simultaneously purchased $\frac{a_2}{a_2-K}$ units of the option (for free) and produced $(a_1, 0)$ at a cost of b_1a_1, he or she would effectively replicate, via production and option purchases, the original asset, (a_1, a_2) at a cost of $b_1a_1<1$. The manufactured asset could then resold in the market to realize a strictly positive profit. The option price, which would eliminate this profit, would be

$$(1-b_1a_1)\left(1-\frac{K}{a_2}\right)>0.$$

Thus, neither the upper nor the lower arbitrage bound could prevail in any reasonable equilibrium.

This is the key observation of this paper. Recognizing the opportunities for potential arbitrages between financial markets and physical production technologies, even though they be stochastic, will typically permit tighter no-arbitrage bounds to be placed on assets that do not lie in the span of the market. It is important to emphasize that these narrower bounds emerge without any appeal to consumption-based models of asset pricing.

The state-claims prices that are consistent with a finite (zero) present-value profit for this production technology are given by the points dominated by the marginal cost vector $\mathbf{b}$

$$\{\mathbf{q}:\mathbf{q}\leq\mathbf{b}\}.$$

Our example considers a case where this set intersects the set of no-arbitrage prices. It is easy to think of cases, however, where they may not. However, these are not always of interest economically.

Suppose, for example, that the no-production-arbitrage set of state-claim price lies everywhere below the line segment giving the no-arbitrage prices so that the two sets do not intersect. Such a situation cannot prevail in any reasonable equilibrium because it implies that the asset (a_1, a_2) could always be constructed for a cost less than its selling price of one. This is a money pump, at least at the prevailing input and asset prices. Thus, the arbitrage opportunity would be eliminated by the asset price and the input prices adjusting.

2 Notation and Preliminaries

Denote the unit vector by $\mathbf{1} \in \Re^S_+$. For a convex function [1] $f : \Re^S \to \Re$, its *subdifferential* at $\mathbf{m}$ is the closed, convex set:

$$\partial f(\mathbf{m}) = \left\{\mathbf{k} \in \Re^S : f(\mathbf{m}) + \mathbf{k}(\mathbf{m}' - \mathbf{m}) \leq f(\mathbf{m}') \text{ for all } \mathbf{m}'\right\}. \tag{1}$$

The elements of $\partial f(\mathbf{m})$ are referred to as *subgradients.* The *one-sided directional (Gateaux) derivative* of f in the direction of $\mathbf{n}$ is defined by

$$f'(\mathbf{m}; \mathbf{n}) = \lim_{\lambda \to 0^+} \left\{\frac{f(\mathbf{m} + \lambda \mathbf{n}) - f(\mathbf{m})}{\lambda}\right\}.$$

For f convex, $f'(\mathbf{m}; \mathbf{n})$ is positively linearly homogeneous and convex (sublinear) in $\mathbf{n}$. So long as f is finite, f' exists, and

$$f'(\mathbf{m}; \mathbf{n}) = \sup\{\mathbf{kn} : \mathbf{k} \in \partial f(\mathbf{m})\}. \tag{2}$$

Therefore,

$$f'(\mathbf{m}; \mathbf{n}) \geq -f'(\mathbf{m}; -\mathbf{n}).$$

When $f'(\mathbf{m}; \mathbf{n}) = -f'(\mathbf{m}; -\mathbf{n})$, we say that f is *smooth in the direction of* $\mathbf{n}$ *at* $\mathbf{m}$. When f is smooth in all directions at $\mathbf{m}$, it is *differentiable.* Moreover, if f is differentiable at $\mathbf{m}$, $\partial f(\mathbf{m})$ is a singleton and corresponds to the usual gradient. If $\partial f(\mathbf{m})$ is a singleton, f is differentiable at $\mathbf{m}$.

The convex conjugate of f is denoted by the convex function, f^*,

$$f^*(\mathbf{k}) = \sup_{\mathbf{m}} \{\mathbf{km} - f(\mathbf{m})\}.$$

[1] Apart from our notion of smoothness in a particular direction, these results on convex functions are all drawn directly from Rockafellar (1970).

If f is proper and closed,[2] then f^* is proper and closed and

$$f(\mathbf{m}) = \sup_{\mathbf{k}} \{\mathbf{km} - f^*(\mathbf{k})\}, \tag{3}$$

and on the relative interior of their domains

$$\mathbf{k} \in \partial f(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial f^*(\mathbf{k}). \tag{4}$$

3 State-Contingent Technologies and the Asset Structure

We model a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty is resolved by 'Nature' making a choice from $\Omega = \{1, 2, ..., S\}$. Each element of Ω is referred to as a state of nature.

There exists a stochastic production technology represented by a single-product, state-contingent input correspondence. Let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \Re_+^S$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0. If state $s \in \Omega$ is realized (picked by 'Nature'), and the *ex ante* input–output combination $(\mathbf{x}, \mathbf{z})$ is chosen, the realized or *ex post* output in period 1 is z_s.

The continuous input correspondence, $X : \Re_+^S \to \Re_+^N$, maps state-contingent output vectors into input sets that are capable of producing them:

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

In addition to continuity, we impose the following properties on $X(\mathbf{z})$:

X.1 $X(\mathbf{0}_{MxS}) = \Re_+^N$ (no fixed costs), and $\mathbf{0} \notin X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$ (no free lunch).

X.2 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.3 $\lambda X(\mathbf{z}) + (1-\lambda)X(\mathbf{z}') \subseteq X(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \qquad 0 \leq \lambda \leq 1$.

X. 4 $X(\mu\mathbf{z}) = \mu X(\mathbf{z})$, $\mu > 0$.

[2] f is proper if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$, and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$. A proper convex function is closed if is lower-semicontinuous.

The first part of X.1 says that doing nothing is always feasible, while the second part of X.1 says that realizing a positive output in any state of nature requires the commitment of some inputs. X.2 says that if an input combination can produce a particular mix of state-contingent outputs then it can always be used to produce a smaller mix of state-contingent outputs. X.3 ensures that the graph of the input correspondence

$$T = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in X(\mathbf{z})\}$$

is convex implying that the technology exhibits diminishing marginal returns in its usual sense. X. 4 implies that the technology exhibits constant returns to scale.

Period 0 input prices are denoted by $\mathbf{w} \in \Re_{+}^{N}$ and are non-stochastic. Financial markets are frictionless, and the *ex ante* financial security payoffs (measured in the same units as $\mathbf{z}$) are given by the $S \times J$ non-negative matrix $\mathbf{A}$. The vector of state-contingent payoffs on the jth financial asset is denoted $\mathbf{A}_j \in \Re_{+}^{S}$, and its price is denoted v_j. The portfolio vector, corresponding to the period 0 purchases of the financial assets, is denoted $\mathbf{h} \in \Re^{J}$. With little true loss of generality, we assume that $\mathbf{A}$ is of full column rank so that there are no redundant assets. Denote the span of the financial markets by $M \subset \Re^{S}$,

$$M = \left\{\mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{h}, \mathbf{h} \in \Re^{J}\right\}.$$

3.1 Production cost structure

Dual to $X(\mathbf{z})$ is the production cost function, $c : \Re_{+}^{N} \times \Re_{+}^{S} \rightarrow \Re_{+}$,

$$c(\mathbf{w}, \mathbf{z}) = \min_{\mathbf{x}}\{\mathbf{w}\mathbf{x} : \mathbf{x} \in X(\mathbf{z})\} \quad \mathbf{w} \in \Re_{+}^{N}$$

if there exists an $\mathbf{x} \in X(\mathbf{z})$ and ∞ otherwise. $c(\mathbf{w}, \mathbf{z})$ is equivalent to the multi-product cost function familiar from non-stochastic production theory (Färe 1988). If the input correspondence satisfies properties X, $c(\mathbf{w}, \mathbf{z})$ satisfies (Chambers and Quiggin, 2000): $c(\mathbf{w}, \mathbf{0}) = 0$; $\mathbf{z}^{o} \geq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}^{o}) \geq c(\mathbf{w}, \mathbf{z})$; $c(\mathbf{w}, \mathbf{z})$ is convex on $\Re_{+}^{S}$ and continuous on the interior of the region where it is finite, and $c(\mathbf{w}, \mu\mathbf{z}) = \mu c(\mathbf{w}, \mathbf{z})$, $\mu > 0$.[3] Thus, c is

[3] Because it is a cost function, c, also satisfies monotonicity and curvature properties in $\mathbf{w}$. We do not use these in what follows, and they are therefore not discussed. Chambers and Quiggin (2000) contains a complete discussion.

sublinear in $\mathbf{z}$.[4]

Let $\mathbf{q} \in \Re_{+}^{S}$ denote a vector of present-value prices (state-claim prices, a stochastic discount factor). The convex conjugate of c,

$$c^{*}(\mathbf{w}, \mathbf{q}) = \sup_{\mathbf{z}} \{\mathbf{qz} - c(\mathbf{w}, \mathbf{z})\},$$

is the present-value profit-function for the present-value prices $\mathbf{q}$. Because c is sublinear in $\mathbf{z}$, $c^{*}(\mathbf{w}, \mathbf{q})$ equals either 0 or ∞. Let $\mathbf{z}' \in \arg\sup \{\mathbf{qz} - c(\mathbf{w}, \mathbf{z})\}$, then

$$\mathbf{qz}' - c(\mathbf{w}, \mathbf{z}') \geq \mathbf{qz} - c(\mathbf{w}, \mathbf{z})$$

for all $\mathbf{z}$, and hence $\mathbf{q} \in \partial c(\mathbf{w}, \mathbf{z}')$. By (4), we then obtain $\mathbf{z}' \in \partial c^{*}(\mathbf{w}, \mathbf{q})$, which restates Hotelling's lemma in terms of subdifferentials. Denote

$$P(\mathbf{w}) = \{\mathbf{q} : c^{*}(\mathbf{w}, \mathbf{q}) = 0\}.$$

3.2 Asset valuation in financial markets

We now restate some asset-pricing results in a dual fashion that presages our theoretical development of the tighter arbitrage bounds. Dual to $\mathbf{A}$ is the *valuation functional* (for example, Prisman, 1986; and Ross, 1987) for $\mathbf{r}$, $p : \Re_{+}^{J} \times \Re_{+}^{S} \to \Re$, defined by the linear program

$$p(\mathbf{v}, \mathbf{r}) = \min \{\mathbf{vh} : \mathbf{Ah} \geq \mathbf{r}\},$$

if $\{\mathbf{h} : \mathbf{Ah} \geq \mathbf{y}\}$ is nonempty and ∞ otherwise.

p has various interpretations. It is the price of the cheapest portfolio that dominates $\mathbf{r}$. But, it is also the (minimum) cost function associated with the linear technology, $\mathbf{A}$. Its basic properties, therefore, are straightforward consequences of standard results on cost functions for linear technologies. It is sublinear and nondecreasing in $\mathbf{r}$ and $p(\mathbf{v}, \mathbf{0}) \leq 0$. Because p is nondecreasing in $\mathbf{r}$, if it exists $\partial_{\mathbf{r}} p(\mathbf{v}, \mathbf{r}) \subset \Re_{+}^{S}$. [5] Perhaps less familiarly, if $\mathbf{r}$ is

[4] A function is sublinear if it is both subadditive and positively linearly homogeneous.

[5] p also has the properties of a cost function in $\mathbf{v}$. Because these are not crucial to our analysis, they are not discussed. Chambers and Quiggin (2003) contains a thorough discussion.

translated in the direction of any of the basic financial assets, its value increases by exactly the asset price times the length of the translation. More formally,

$$\begin{aligned} p(\mathbf{v},\mathbf{r}+\delta\mathbf{A}_j) &= \min\{\mathbf{vh}:\mathbf{Ah}\geq\mathbf{r}+\delta\mathbf{A}_j\} \\ &= \min\left\{\mathbf{v}_{-j}\mathbf{h}_{-j}+v_j(h_j-\delta):\mathbf{A}_{-\mathbf{j}}\mathbf{h}_{-j}+(h_j-\delta)\mathbf{A}_j\geq\mathbf{r}\right\}+v_j\delta \\ &= p(\mathbf{v},\mathbf{r})+\delta v_j, \quad \delta\in\Re. \end{aligned} \tag{5}$$

Expression (5) implies $p'\left(\mathbf{v},\mathbf{r},\mathbf{A}_j\right)=v_j=-p'\left(\mathbf{v},\mathbf{r},-\mathbf{A}_j\right)$ so that $p(\mathbf{v},\mathbf{r})$ is smooth in the direction of any of the basic assets.

The traditional notion of the absence of a financial arbitrage can be defined dually in terms of $p(\mathbf{v},\mathbf{r})$ (Prisman, 1986; Ross, 1987). An arbitrage exists if there is either a zero-priced portfolio for which $\mathbf{r}\geq\mathbf{0}$ but $\mathbf{r}\neq\mathbf{0}$, or if there is a negatively priced portfolio for which $\mathbf{r}=\mathbf{0}$ (Ross, 1978; Prisman, 1986; Ross, 1987; Magill and Quinzii, 1995; LeRoy and Werner, 2000). Thus, the absence of arbitrage requires $p(\mathbf{v},\mathbf{r})>0$ for $\mathbf{r}\geq\mathbf{0}$ with $\mathbf{r}\neq\mathbf{0}$ and $p(\mathbf{v},\mathbf{0})\geq 0$.

By the basic properties of p, $p(\mathbf{v},\mathbf{0})\leq 0$. Together with the requirement for the absence of arbitrage that $p(\mathbf{v},\mathbf{0})\geq 0$, this establishes that *the absence of an arbitrage implies* $p(\mathbf{v},\mathbf{0})=0$.

The convex conjugate of p is the present-value arbitrage profit function [6] defined by

$$p^*(\mathbf{v},\mathbf{q})=\sup_{\mathbf{r}}\{\mathbf{qr}-p(\mathbf{v},\mathbf{r})\}.$$

If $\mathbf{q}'\in\arg\sup_{\mathbf{q}}\{\mathbf{qr}-p^*(\mathbf{v},\mathbf{q})\}$, then

$$\mathbf{q}'\mathbf{r}-p^*(\mathbf{v},\mathbf{q}')\geq\mathbf{qr}-p^*(\mathbf{v},\mathbf{q})$$

for all $\mathbf{q}$, and thus

$$\mathbf{r}\in\partial_{\mathbf{q}}p^*(\mathbf{v},\mathbf{q}'). \tag{6}$$

Because p is sublinear over $\mathbf{r}$, p^* equals either 0 or ∞. Therefore, in the absence of

[6] Prisman (1986) refers to $p^*(\mathbf{v},\mathbf{q})$ as the indirect arbitrage function.

arbitrage, the conjugacy between $p(\mathbf{v},\mathbf{0})$ and $p^*(\mathbf{v},\mathbf{q})$ implies

$$\begin{aligned} p(\mathbf{v},\mathbf{0}) &= \sup_{\mathbf{q}} \{-p^*(\mathbf{v},\mathbf{q})\} \\ &= -\inf_{\mathbf{q}} \{p^*(\mathbf{v},\mathbf{q})\} \\ &= 0. \end{aligned}$$

Thus, the absence of arbitrage implies the existence of a set of state-claim prices

$$\mathcal{N}(\mathbf{v}) = \{\mathbf{q} : p^*(\mathbf{v},\mathbf{q}) = 0\}.$$

Using (6) and (4) gives

$$\begin{aligned} \mathcal{N}(\mathbf{v}) &= \partial_{\mathbf{r}} p(\mathbf{v},\mathbf{0}) \\ &= \left\{\mathbf{q} \in \Re^S_+ : \mathbf{qA} = \mathbf{v}\right\}. \end{aligned}$$

$\mathcal{N}(\mathbf{v})$ is the set of no-arbitrage prices.

The upper no-arbitrage bound on the price of $\mathbf{r}$ is the maximal valuation of $\mathbf{r}$ over $\mathcal{N}(\mathbf{v})$ (LeRoy and Werner, 2000; Cochrane, 2001). Denoting this upper bound by $u(\mathbf{r})$, we have

$$\begin{aligned} u(\mathbf{r}) &= \sup\{\mathbf{qr} : \mathbf{q} \in \mathcal{N}(\mathbf{v})\} \\ &= \sup\{\mathbf{qr} : \mathbf{q} \in \partial_{\mathbf{r}} p(\mathbf{v},\mathbf{0})\} \\ &= p'(\mathbf{v},\mathbf{0};\mathbf{r}). \end{aligned} \tag{7}$$

The upper arbitrage bound on $\mathbf{r}$, therefore, equals the directional derivative of $p(\mathbf{v},\mathbf{0})$ in the direction $\mathbf{r}$, and thus $u(\mathbf{r})$ is the marginal cost from an initial position of $\mathbf{0}$ required to accomodate an $\mathbf{r}$ payoff. This yields its frequent interpretation as the maximal buying price of $\mathbf{r}$.

Symmetrically, the lower-arbitrage bound on the asset price is

$$\begin{aligned} l(\mathbf{r}) &= \inf\{\mathbf{qr} : \mathbf{q} \in \mathcal{N}(\mathbf{v})\} \\ &= -\sup\{-\mathbf{qr} : \mathbf{q} \in \partial_{\mathbf{r}} p(\mathbf{v},\mathbf{0})\} \\ &= -p'(\mathbf{v},\mathbf{0};-\mathbf{r}). \end{aligned}$$

Hence, the lower bound on $\mathbf{r}$ is minus the marginal cost from an initial position of $\mathbf{0}$ required to accomodate a payout of $\mathbf{r}$. Thus, its frequent interpretation as the minimal selling price of $\mathbf{r}$.

By the sublinearity of directional derivatives, $p'(\mathbf{v}, \mathbf{0}; \mathbf{r}) \geq -p'(\mathbf{v}, \mathbf{0}; -\mathbf{r})$ with a strict inequality if p is not smooth in the direction of $\mathbf{r}$. Because $p(\mathbf{v}, \mathbf{0})$ is not always differentiable,[7] there can exist a gap between $u(\mathbf{r})$ and $l(\mathbf{r})$.

However, $p(\mathbf{v}, \mathbf{0})$ is smooth, and therefore $u(\hat{\mathbf{r}}) = l(\hat{\mathbf{r}})$, in any direction $\hat{\mathbf{r}} \in M$. This can be seen in several ways. For example, smoothness in such directions follows from a repeated application of the implication from (5) that $p'\left(\mathbf{v}, \mathbf{0}; \mathbf{A}_j\right) = v_j$. Alternatively, it is a consequence the linearity of p over M (e.g., Clark, 1993; LeRoy and Werner, 2000). Or, by construction for any $\hat{\mathbf{r}} \in M$[8]

$$\begin{aligned} p(\mathbf{v}, \hat{\mathbf{r}}) &= \mathbf{v}\hat{\mathbf{h}} \\ &= \mathbf{v}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\hat{\mathbf{r}} \\ &= \bar{\mathbf{q}}(\mathbf{v})\hat{\mathbf{r}}. \end{aligned}$$

We refer to $\bar{\mathbf{q}}(\mathbf{v})$ as the *pricing kernel* for M.[9] It is the unique orthogonal projection of $\mathcal{N}(\mathbf{v})$ onto M.

We conclude this section by noting yet another means of representing the no-arbitrage bounds in terms of p.[10]

Theorem 1

$$\begin{aligned} u(\mathbf{r}) &= p(\mathbf{v}, \mathbf{r}), \\ l(\mathbf{r}) &= -p(\mathbf{v}, -\mathbf{r}). \end{aligned}$$

[7]An exception occurs, for example, if markets are complete.

[8]Recall, by assumption, $\mathbf{A}$ contains only basis assets so that it is of full column rank.

[9]It is known variously as the mimicking portfolio, the ideal portfolio, and the ideal discount factor.

[10]That $u(\mathbf{r}) = p(\mathbf{v}, \mathbf{r})$ can also be seen by considering the dual program to the linear program defining $p(\mathbf{v}, \mathbf{r})$.

Proof The proof is for $u(\mathbf{r})$, the proof for $l(\mathbf{r})$ is symmetric. By definition

$$\begin{aligned} p'(\mathbf{v},\mathbf{0};\mathbf{r}) &= \lim_{\lambda\to 0^+}\left\{\frac{p(\mathbf{v},\lambda\mathbf{r}) - p(\mathbf{v},\mathbf{0})}{\lambda}\right\} \\ &= \lim_{\lambda\to 0^+}\left\{\frac{p(\mathbf{v},\lambda\mathbf{r})}{\lambda}\right\} \\ &= p(\mathbf{v},\mathbf{r}). \end{aligned}$$

The second equality follows by the no-arbitrage requirement that $p(\mathbf{v},\mathbf{0}) = 0$, and the third follows by the sublinearity of p in $\mathbf{r}$. Apply (7).■

4 Tighter Bounds

Our example demonstrates that the bounds $u(\mathbf{r})$ and $l(\mathbf{r})$ may be consistent with asset prices for which arbitrages exist, once physical production opportunities have been taken into account. This section details how the the assumption of the absence of production and financial arbitrages can be used to narrow the no-arbitrage bounds implied by $u(\mathbf{r})$ and $l(\mathbf{r})$. The critical analytic concept is the derivative-cost function to which we now turn.

4.1 The derivative-cost function

Define the *derivative-cost function* $C : \Re^N_+ \times \Re^J_+ \times \Re^S \to \Re$, by

$$C(\mathbf{w},\mathbf{v},\mathbf{y}) = \inf_{\mathbf{r},\mathbf{z}}\{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \mathbf{r}+\mathbf{z} \geq \mathbf{y}\},$$

C is the cost of the cheapest asset and production portfolio that dominates $\mathbf{y} \in \Re^S$. We establish in an appendix:[11]

Theorem 2 *C satisfies:*

1. $C(\mathbf{w},\mathbf{v},\mathbf{y})$ is a nondecreasing, sublinear function of $\mathbf{y}$ that is continuous on the interior of the region where it is finite.

[11] C is a cost function, and thus it possesses standard properties in terms of the input prices $(\mathbf{v},\mathbf{w})$. These are not germane to the central part of our argument and are thus not considered. Chambers and Quiggin (2003) contains a more detailed discussion.

2. $C(\mathbf{w}, \mathbf{v}, \mathbf{0}) \leq 0.$

3. $C\left(\mathbf{w}, \mathbf{v}, \mathbf{y}+\boldsymbol{\delta}\mathbf{A}_j\right) = C(\mathbf{w}, \mathbf{v}, \mathbf{y}) + \delta v_j \quad \mathbf{y}+\boldsymbol{\delta}\mathbf{A}_j \in \Re_+^S.$

Given present-value prices, $\mathbf{q} \in \Re_+^S$ the convex conjugate of C,

$$C^*(\mathbf{w}, \mathbf{v}, \mathbf{q}) = \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})\},$$

is the present-value profit function derivable from simultaneous access to the physical production technology and financial markets. Because $C(\mathbf{w}, \mathbf{v}, \mathbf{y})$ is sublinear in $\mathbf{y}$, $C^*(\mathbf{w}, \mathbf{v}, \mathbf{q})$ is either 0 or ∞.

For any $\mathbf{y}' \in \arg\sup \{\mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})\}$,

$$\mathbf{q}\mathbf{y}' - C(\mathbf{w}, \mathbf{v}, \mathbf{y}') \geq \mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})$$

for all $\mathbf{y}$. Thus,

$$\mathbf{q} \in \partial_{\mathbf{y}} C(\mathbf{w}, \mathbf{v}, \mathbf{y}'). \tag{8}$$

Applying (4) to (8) establishes

$$\mathbf{y}' \in \partial_{\mathbf{q}} C^*(\mathbf{w}, \mathbf{v}, \mathbf{q}). \tag{9}$$

Expression (9) restates Hotelling's Lemma for the current problem.

A closer examination of the structure of C^* proves beneficial. For $\mathbf{q} \in \Re_+^S$,

$$\begin{aligned}
C^*(\mathbf{w}, \mathbf{v}, \mathbf{q}) &= \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})\} \\
&= \sup_{\mathbf{y}} \left\{\mathbf{q}\mathbf{y} - \min_{\mathbf{r},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + p(\mathbf{v}, \mathbf{r}) : \mathbf{r} + \mathbf{z} \geq \mathbf{y}\}\right\} \\
&= \sup_{\mathbf{y},\mathbf{r},\mathbf{z}} \{\mathbf{q}\mathbf{y} - \mathbf{c}(\mathbf{w}, \mathbf{z}) - p(\mathbf{v}, \mathbf{r}) : \mathbf{r} + \mathbf{z} \geq \mathbf{y}\} \\
&= \sup_{\mathbf{r},\mathbf{z}} \{\mathbf{q}(\mathbf{r} + \mathbf{z}) - \mathbf{c}(\mathbf{w}, \mathbf{z}) - p(\mathbf{v}, \mathbf{r})\} \\
&= c^*(\mathbf{w}, \mathbf{q}) + p^*(\mathbf{v}, \mathbf{q}) \\
&= \begin{cases} \infty & \mathbf{q} \notin \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w}) \\ 0 & \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w}) \end{cases}.
\end{aligned} \tag{10}$$

Applying the conjugacy (3) relation to (10) yields an important structural result for C:

Theorem 3 *If* $C(\mathbf{w},\mathbf{v},\mathbf{y}) > -\infty$, *then*

$$\begin{aligned} C(\mathbf{w},\mathbf{v},\mathbf{y}) &= \sup_{\mathbf{q}} \{\mathbf{qy} - c^*(\mathbf{w},\mathbf{q}) : \mathbf{q} \in \mathcal{N}(\mathbf{v})\} \\ &= \sup_{\mathbf{q}} \{\mathbf{qy} - p^*(\mathbf{v},\mathbf{q}) : \mathbf{q} \in P(\mathbf{w})\} \\ &= \sup_{\mathbf{q}} \{\mathbf{qy} : \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})\}. \end{aligned}$$

4.2 The No-Arbitrage Bounds

To develop tighter bounds, it is necessary to extend the notion of a financial arbitrage to include the type of arbitrages noted in our earlier example. We now say that *an arbitrage exists* if there exists an asset portfolio, $\mathbf{h}$, and a technically feasible input-output combination, $(\mathbf{x},\mathbf{z})$, for which either $\mathbf{vh}+\mathbf{wx} = 0$ and $\mathbf{Ah}+\mathbf{z} \geq \mathbf{0}$, $\mathbf{Ah}+\mathbf{z} \neq \mathbf{0}$, or $\mathbf{vh}+\mathbf{wx} < 0$ and $\mathbf{Ah}+\mathbf{z} = \mathbf{0}$. Expressed in terms of C, the *absence of an arbitrage* requires that there exist no $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ for which $C(\mathbf{w},\mathbf{v},\mathbf{y}) = 0$, and that $C(\mathbf{w},\mathbf{v},\mathbf{0}) \geq 0$. When combined with Theorem 2.2, *the absence of arbitrage, therefore, requires that* $C(\mathbf{w},\mathbf{v},\mathbf{0}) = 0$.

Theorem 3 and the absence of arbitrage imply

$$\begin{aligned} C(\mathbf{w},\mathbf{v},\mathbf{0}) &= \sup_{\mathbf{q}} \{\mathbf{q0} : \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})\} \\ &= 0. \end{aligned}$$

Thus, the state-claim prices that rule out an arbitrage correspond to

$$\mathcal{N}(\mathbf{v}) \cap P(\mathbf{w}) = \partial_{\mathbf{y}} C(\mathbf{w},\mathbf{v},\mathbf{0}) \subset \Re^S_+,$$

where the equality follows by (4).

We now redefine the arbitrage bounds. The *upper no-arbitrage bound* on a $\mathbf{y}$ is its maximal valuation over the no-arbitrage prices $\mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$. Denoting this upper bound by $U(\mathbf{y})$,

$$\begin{aligned} U(\mathbf{y}) &= \sup \{\mathbf{qy} : \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})\} \\ &= \sup \{\mathbf{qy} : \mathbf{q} \in \partial_{\mathbf{y}} C(\mathbf{w},\mathbf{v},\mathbf{0})\} \\ &= C'(\mathbf{w},\mathbf{v},\mathbf{0};\mathbf{y}) \\ &= C(\mathbf{w},\mathbf{v},\mathbf{y}), \end{aligned}$$

where the third equality follows by basic properties of directional derivatives, and the fourth now follows from Theorem 3.

Symmetrically, the *lower no-arbitrage bound* is

$$\begin{aligned} L(\mathbf{y}) &= -C'(\mathbf{w}, \mathbf{v}, \mathbf{0}; -\mathbf{y}) \\ &= -C(\mathbf{w}, \mathbf{v}, -\mathbf{y}), \end{aligned}$$

where the second equality is established exactly as in the proof of Theorem 1.

These observations yield our central result:

Theorem 4 $u(\mathbf{y}) = p(\mathbf{v}, \mathbf{y}) \geq U(\mathbf{y}) = C(\mathbf{w}, \mathbf{v}, \mathbf{y}) \geq L(\mathbf{y}) = -C(\mathbf{w}, \mathbf{v}, -\mathbf{y}) \geq l(\mathbf{y}) = -p(\mathbf{v}, -\mathbf{y})$.

Proof $U(\mathbf{y}) \geq L(\mathbf{y})$ follows from the sublinearity of directional deriviatives, $C'(\mathbf{w}, \mathbf{v}, \mathbf{0}; \mathbf{y}) \geq -C'(\mathbf{w}, \mathbf{v}, -\mathbf{y})$. $u(\mathbf{y}) \geq U(\mathbf{y})$ follows because

$$\begin{aligned} u(\mathbf{y}) &= \sup\{\mathbf{qy} : \mathbf{q} \in \mathcal{N}(\mathbf{v})\} \\ &\geq \sup\{\mathbf{qy} : \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})\} \\ &= U(\mathbf{y}), \end{aligned}$$

and $L(\mathbf{y}) \geq l(\mathbf{y})$ follows symmetrically. The equalities are demonstrated in the text.■

4.3 Unique no-arbitrage prices for nonreplicable assets

p is linear over M (Clark, 1993). If $\mathbf{r} \in M$, then $-\mathbf{r} \in M$, and thus trivially

$$\begin{aligned} u(\mathbf{r}) &= p(\mathbf{v}, \mathbf{r}) \\ &= \bar{\mathbf{q}}(\mathbf{v})\mathbf{r} \\ &= -\bar{\mathbf{q}}(\mathbf{v})(-\mathbf{r}) \\ &= -p(\mathbf{v}, -\mathbf{r}) \\ &= l(\mathbf{r}). \end{aligned}$$

Assets in M can be uniquely priced by $\bar{\mathbf{q}}(\mathbf{v})$.Thus, the upper and lower no-arbitrage bounds coincide. Naturally, a similar property is inherited by $U(\mathbf{y})$ and $L(\mathbf{y})$. If $\mathbf{y} \in M$, then Theorem 2.3 implies that $U(\mathbf{y}) = L(\mathbf{y})$. However, the properties of $C(\mathbf{w}, \mathbf{v}, \mathbf{y})$ allow us to strengthen this result and establish that *some assets outside the span of the market can by be uniquely priced using our no-arbitrage pricing techniques.* Our next theorem details conditions under which this applies.

Theorem 5 *If* $\bar{\mathbf{q}}(\mathbf{v}) \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$, *then* $\mathbf{y} \in \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + M$ *and* $-\mathbf{y} \in \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + M$ *if and only if* $U(\mathbf{y}) = \bar{\mathbf{q}}(\mathbf{v})\mathbf{y} = L(\mathbf{y})$.

Proof Suppose that $U(\mathbf{y}) = \bar{\mathbf{q}}(\mathbf{v})\mathbf{y} = L(\mathbf{y})$. Theorem 4 and Theorem 3 imply

$$\begin{aligned} C(\mathbf{w}, \mathbf{v}, \mathbf{y}) &= \bar{\mathbf{q}}(\mathbf{v})\mathbf{y}, \\ C(\mathbf{w}, \mathbf{v}, -\mathbf{y}) &= -\bar{\mathbf{q}}(\mathbf{v})\mathbf{y}. \end{aligned}$$

Thus

$$\begin{aligned} \bar{\mathbf{q}}(\mathbf{v}) &\in \partial_{\mathbf{y}} C(\mathbf{w}, \mathbf{v}, \mathbf{y}), \\ \bar{\mathbf{q}}(\mathbf{v}) &\in \partial_{\mathbf{y}} C(\mathbf{w}, \mathbf{v}, -\mathbf{y}). \end{aligned}$$

Applying (9) gives

$$\begin{aligned} \mathbf{y} &\in \partial_{\mathbf{q}} C^*(\mathbf{w}, \mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + \partial_{\mathbf{q}} p^*(\mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + M, \end{aligned}$$

where the second equality follows because $\bar{\mathbf{q}}(\mathbf{v}) \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$ implies that the relative interiors of the domains of c^* and p^* share a point in common. Symmetrically,

$$\begin{aligned} -\mathbf{y} &\in \partial_{\mathbf{q}} C^*(\mathbf{w}, \mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + \partial_{\mathbf{q}} p^*(\mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + M. \end{aligned}$$

Conversely, suppose $\mathbf{y} \in \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + M$, then since $M = \partial_{\mathbf{q}} p^*(\mathbf{v}, \bar{\mathbf{q}}(\mathbf{v}))$

$$\begin{aligned} \mathbf{y} &\in \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) + \partial_{\mathbf{q}} p^*(\mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \partial_{\mathbf{q}} C^*(\mathbf{w}, \mathbf{v}, \bar{\mathbf{q}}(\mathbf{v})), \end{aligned}$$

and by (9) $\bar{\mathbf{q}}(\mathbf{v}) \in \partial_{\mathbf{y}} C(\mathbf{w}, \mathbf{v}, \mathbf{y})$, and thus $C(\mathbf{w}, \mathbf{v}, \mathbf{y}) = \bar{\mathbf{q}}(\mathbf{v})\mathbf{y}$. The argument for $-\mathbf{y}$ is symmetric.■

The conditions of Theorem 5 closely match Chambers and Quiggin's (2003) necessary and sufficient conditions for the local separation of production decisions from the producer's risk attitudes. Separation requires that the output decisions of the firm are independent of the producer's attitudes towards risk, and marginal production choices are guided by $\bar{\mathbf{q}}(\mathbf{v})$. Dually, therefore, if $\mathbf{y}$ and $-\mathbf{y}$ are equivalent modulo M to an element of $\partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$, then $\mathbf{y}$ can be priced uniquely by $\bar{\mathbf{q}}(\mathbf{v})$.

5 Relation to Other Work

Even though the approaches of Cochrane and Saá-Requejo and Bernardo and Ledoit are motivated by grafting restrictions developed from relatively robust aspects of consumption-based pricing models onto arbitrage-bound methods, they naturally share a structural similarity with the methods that we have suggested. However, it is important to recognize that both the good-deal and the gain-loss ratio approach are predicated on the existence of a probability measure for Ω. They are thus expressed in terms of expectation inner products instead of the inner products used here. Our analysis requires no assumption on the underlying probability measure for Ω, and thus applies regardless of whether there exist objective probabilities or whether individuals form rational expectations.

The good-deal approach, in terms of the inner products used here, derives the upper bound as

$$\sup_{\mathbf{q} \geq \mathbf{0}} \left\{ \mathbf{q}\mathbf{y} : \mathbf{q}'\mathbf{A} = \mathbf{v}; \sigma(\mathbf{q}) \leq \frac{h}{r^f} \right\},$$

where σ denotes the variance, h is a parameter, and r^f is the risk-free rate. The lower bound is derived symmetrically. We can re-express this bound as

$$\sup_{\mathbf{q} \geq \mathbf{0}} \left\{ \mathbf{q}\mathbf{y} : \mathbf{q}'\mathbf{A} = \mathbf{v}; \sqrt{\sigma(\mathbf{q})} \leq \sqrt{\frac{h}{r^f}} \right\}.$$

$\sqrt{\sigma(\mathbf{q})}$ is a sublinear function of $\mathbf{q}$. Thus, by an appropriate translation (note the standard deviation is translation invariant), $\sqrt{\sigma(\mathbf{q})}$ can always be reinterpreted as the gauge function for a convex set of state-claim price vectors centered at the origin (e.g., Luenberger, 1969). More specifically, this constraint set corresponds to a convex set centered around the payout on the riskless asset. The dimensions of the set (and thus the tightness of the bounds) are determined by the choice of h. The good-deal approach, therefore, narrows the no-arbitrage bounds by restricting choice of the present-value price vector to lie in the intersection of $\mathcal{N}(\mathbf{v})$ and the convex set centered around the payout to the riskless asset.

The gain-loss ratio approach derives the upper bound[12] as

$$\sup_{\mathbf{q}>\mathbf{0}}\left\{\mathbf{q}\mathbf{y}:\mathbf{q}'\mathbf{A}=\mathbf{v};\frac{\sup\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}}{\inf\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}}\leq L\right\}.$$

Here $\mathbf{q}^*\in\Re^S_{++}$ is a 'benchmark' set of state-claim prices deduced from an expected-utility functional.[13] The key theoretical insight of Bernardo and Ledoit (2000) is that $\frac{\sup\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}}{\inf\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}}$ dually reflects a constraint on the ratio of expected gain (the expected-value, calculated over a 'risk-neutral probability' measure, of the positive component of excess returns) to expected loss (the expected value over the negative component of excess returns).

There is, however, yet another dual interpretation of this constraint. Notice that

$$\sup\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}=\sup_{\mathbf{z}\geq 0}\left\{\sum_{s\in\Omega}q_sz_s:\sum_{s\in\Omega}q_s^*z_s=1\right\},$$

and that

$$\inf\left\{\frac{q_1}{q_1^*},...,\frac{q_S}{q_S^*}\right\}=\inf_{\mathbf{z}\geq\mathbf{0}}\left\{\sum_{s\in\Omega}q_sz_s:\sum_{s\in\Omega}q_s^*z_s=1\right\}.$$

The numerator, therefore, corresponds to the maximal valuation (in terms of $\mathbf{q}$) over the set of non-negative assets priced at one by $\mathbf{q}^*$.[14] The denominator is the corresponding minimal valuation. Thus, the dual gain-loss ratio restricts this maximal valuation to be no more than L times greater than the minimal valuation.

[12] The gain-loss approach in Bernardo and Ledoit (2000) is expressed in terms of excess payout space. So as to not introduce further notation, we carry on the discussion in payout space.

[13] However, nothing precludes $\mathbf{q}^*$ from being generated by other means. For example, it could be deduced from a representative firm's cost subdifferential.

[14] More formally, the numerator is the maximal support function for this set.

In this notation, our analysis derives

$$U(\mathbf{y}) = \sup_{\mathbf{q}>\mathbf{0}} \{\mathbf{qy} : \mathbf{q}'\mathbf{A} = \mathbf{v}; c^*(\mathbf{w},\mathbf{q}) = 0\}.$$

Thus, our tightened upper bound differs from the good-deal and gain-loss upper bound by the extra constraint introduced into the mathematical programming problem defining the upper bound. Our extra constraint emerges not from restrictions on either preferences or maximal and minimal valuations, but from the need to exclude arbitrages associated with the stochastic technology. Thus, the additional constraint requires that the state-claim prices fall on the convex, zero-profit isoprofit contour.

In our example, we depict the convenient polar case (linear cost structure) where the isoprofit contour assumes a fixed coefficient form. At the other extreme (fixed coefficient cost structure) is a linear present-value profit function with flat isoprofit contours. It is of particular interest to note that a stochastic production function representation of $X(\mathbf{z})$ with a multiplicative productivity shock (Diamond, 1967; Jermann, 1998; Tallarini, 2000),

$$X(\mathbf{z}) = \cap_{s=1}^{S} \{\mathbf{x} : f(\mathbf{x}) a_s \geq z_s\},$$

with f positively linearly homogeneous and concave and $a_s > 0$, yields the other polar isoprofit structure. For this technology

$$c(\mathbf{w},\mathbf{z}) = \gamma(\mathbf{w}) \max\left\{\frac{z_1}{a_1}, ..., \frac{z_S}{a_S}\right\}$$

where $\gamma(\mathbf{w})$ is the cost function dual to the nonstochastic part of the technology $f(\mathbf{x})$. In any optimum,

$$\frac{z_1}{z_s} = \frac{a_1}{a_s},$$

for all s. Present value profit is, is thus, $\frac{z_1}{a_1}(\mathbf{qa} - \gamma(\mathbf{w}))$. Thus,

$$P(\mathbf{w}) = \left\{\mathbf{q} \in \Re_+^S : \mathbf{qa} \leq \gamma(\mathbf{w})\right\}.$$

More generally, one expects a smooth constraint set associated with a smooth convex (to the origin) isoprofit contour.

It follows trivially that this upper bound (and the corresponding lower bound) can be at least weakly tightened by, for example, introducing the good-deal bound constraint to obtain

$$U(\mathbf{y}) \geq U^{g}(\mathbf{y}) = \sup_{\mathbf{q} \geq \mathbf{0}} \left\{ \mathbf{qy} : \mathbf{q}'\mathbf{A} = \mathbf{v}; c^{*}(\mathbf{w}, \mathbf{q}) = 0; \sqrt{\sigma(\mathbf{q})} \leq \sqrt{\frac{h}{r^{f}}} \right\}.$$

Alternatively, one can view our approach as offering a method for further tightening either the good-deal bounds or the gain-loss bounds. For example, the gain-loss ratio upper bound can be tightened further by introducing the production arbitrage constraint into it as

$$\sup_{\mathbf{q} > \mathbf{0}} \left\{ \mathbf{qy} : \mathbf{q}'\mathbf{A} = \mathbf{v}; \frac{\sup\left\{\frac{q_1}{q_1^*}, \ldots, \frac{q_S}{q_S^*}\right\}}{\inf\left\{\frac{q_1}{q_1^*}, \ldots, \frac{q_S}{q_S^*}\right\}} \leq L; c^{*}(\mathbf{w}, \mathbf{q}) = 0 \right\}.$$

Analytically, our analysis is closely related to the contributions of Prisman (1986) and Ross (1987) and to a lesser extent that of Jouini and Kallal (1995). Although their terminology is slightly different, Prisman and Ross use convex analysis to analyze the effect of introducing frictions in the form of convex tax structures into a frictionless financial market. One alternative interpretation of the stochastic technology, because of its nonlinearity, is as a source of friction in assembling state-contingent claims. It is apparent, therefore, that with relatively few changes our arguments can accomodate convex transactions costs and convex tax structures.

Suppose, for example, that investment in asset j incurred, in addition, to its acquistion cost of v_j an adjustment or transactions cost convex in the level of holding $t_j(h_j)$. The corresponding reformulation of the derivative-cost problem would be

$$\min_{\mathbf{h}, \mathbf{z}} \left\{ c(\mathbf{w}, \mathbf{z}) + \mathbf{vh} + \sum_{j} t_j(h_j) : \mathbf{Ah} + \mathbf{z} \geq \mathbf{y} \right\}.$$

This is a convex minimization problem subject to a convex constraint set. Thus, modified versions of our conjugate dual arguments could be used to deduce a conjugate representation of this derivative-cost function and analogues of Theorems 2 and 3.

Alternatively, one can represent a tax code as a mapping, $g : \Re^{S} \to \Re^{S}$, from state-contingent income space into itself. The corresponding reformulation of the derivative-cost problem is

$$C(\mathbf{w},\mathbf{v},\mathbf{y}) = \min_{\mathbf{h},\mathbf{z}} \{c(\mathbf{w},\mathbf{z}) + \mathbf{vh} : \mathbf{Ah} + \mathbf{z} - g(\mathbf{Ah} + \mathbf{z}) \geq \mathbf{y}\}.$$

If g is convex, this remains a convex minimization problem subject to a convex constraint set.

$C(\mathbf{w},\mathbf{v},\mathbf{y})$, as derived in the presence of convex transactions costs or a convex tax structure, is nondecreasing and convex in $\mathbf{y}$. However, it is no longer guaranteed to be positively linearly homogeneous. Moreover, because asset valuation in a market with frictions is individual specific, another notion of an arbitrage is required.[15] Following Prisman (1986), Jouini and Kallal (1995), and Chambers and Quiggin (2002), an arbitrage would exist at an asset holding of $\mathbf{y}$ if there existed a $\mathbf{y}' \geq \mathbf{y}$, $\mathbf{y}' \neq \mathbf{y}$ with $C(\mathbf{w},\mathbf{v},\mathbf{y}') \leq C(\mathbf{w},\mathbf{v},\mathbf{y})$. Such an arbitrage is ruled out if the derivative-cost function is strictly increasing (in place of nondecreasing) in $\mathbf{y}$. The corresponding no-arbitrage prices (at $\mathbf{y}$) are the elements of $\partial C(\mathbf{w},\mathbf{v},\mathbf{y}) \subset \Re^S_{++}$ because

$$\partial C(\mathbf{w},\mathbf{v},\mathbf{y}) = \{\mathbf{q} : \mathbf{qy}' - C(\mathbf{w},\mathbf{v},\mathbf{y}') \leq \mathbf{qy} - C(\mathbf{w},\mathbf{v},\mathbf{y})\}$$

for all $\mathbf{y}$. This subdifferential consists of the prices that rule out any profitable moves from $\mathbf{y}$. The associated upper and lower no-arbitrage bounds, respectively, are

$$\begin{aligned} U(\mathbf{r}) &= C'(\mathbf{w},\mathbf{v},\mathbf{y};\mathbf{r}) = \sup\{\mathbf{qr} : \mathbf{q} \in \partial C(\mathbf{w},\mathbf{v},\mathbf{y})\} \\ &\geq \inf\{\mathbf{qr} : \mathbf{q} \in \partial C(\mathbf{w},\mathbf{v},\mathbf{y})\} \\ &= -C'(\mathbf{w},\mathbf{v},\mathbf{y}; -\mathbf{r}) = L(\mathbf{r}). \end{aligned}$$

In the absence of positive linear homogeneity, these bounds do not correspond to the derivative cost functions as in the present case. However, it follows trivially that the bounds derived in the presence of the technology are at least weakly tighter than the bounds derived in the absence of the technology.

It is also of interest to consider empirical evidence about the usefulness of the proposed approach. Existing statistical evidence links real variables such as investment and

[15] Asset valuation can also be individual specific if different individuals are endowed with different stochastic technologies.

output to asset returns. Depending on the view taken, causality (or perhaps better predictability) can run in either direction. For example, Cochrane's (1991) production-based model uses investment activity and a production function to construct a stochastic discount factor used in the empirical modelling of asset prices. Estrella and Hardouvelis (1991), working in a general-equilibrium framework, derive and estimate a model in which the term structure acts as an indicator of future economic activity. Real-business cycle and general-equilibrium models routinely imbed imbed production structures in their pricing relationships (Jermann, 1998; Tallarini, 2000). Regardless of the direction of causality, the identification of statistical links between the real and financial parts of the economy buttresses the case for extending the no-arbitrage principle beyond the assets that make up the basis.

6 Conclusion

We have demonstrated by example and by analytic argument that the no-arbitrage bounds on assets lying outside the span of the market can potentially by narrowed by ruling out arbitrages between asset markets and stochastic production opportunities. The key analytic construct is the derivative-cost function. The compressed set of no-arbitrage prices implied by the elimination of arbitrage opportunities between asset markets and production opportunities is given by the subdifferential of the derivative-cost function. The narrowed no-arbitrage bounds can be calculated either as directional derivatives of the derivative-cost function at the origin or directly from the derivative-cost function itself. Recognizing possible interactions between the physical production technology and financial markets permits some assets lying outside the subspace generated by the basis assets to be priced uniquely using the no-arbitrage prices. We have characterized how. We have also briefly compared our results to recent contributions that have used information derived from model-based approaches to narrow the no-arbitrage bounds, and we have discussed the relationship between our work and contributions on arbitrage pricing in the presence of market frictions.

7 Appendix: Proof of Theorem

Continuity follows by the theorem of the maximum and the continuity properties of c and p (Berge, 1963). Let $(\mathbf{r}', \mathbf{z}')$ be optimal for $\mathbf{y}' \geq \mathbf{y}$, then C nondecreasing follows because $(\mathbf{r}', \mathbf{z}')$ remains feasible for $\mathbf{y}$. To demonstrate sublinearity, we first demonstrate convexity. Let $(\mathbf{r}', \mathbf{z}')$ and $(\mathbf{r}'', \mathbf{z}'')$ be optimal for $\mathbf{y}'$ and $\mathbf{y}''$, respectively. By the linearity of the constraint sets $(\lambda \mathbf{r}' + (1-\lambda)\mathbf{r}'', \lambda \mathbf{z}' + (1-\lambda)\mathbf{z}'')$ is feasible for $\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}''$. By the convexity of c and p

$$\begin{aligned}
c(\mathbf{w},\lambda\mathbf{z}' + (1-\lambda)\mathbf{z}'') + p(\mathbf{v},\lambda\mathbf{r}' + (1-\lambda)\mathbf{r}'') &\leq \lambda\left[c(\mathbf{w},\mathbf{z}') + p(\mathbf{v},\mathbf{r}')\right] + (1-\lambda)\left[c(\mathbf{w},\mathbf{z}'') + p(\mathbf{v}
\end{aligned}$$
$$\qquad\qquad = \lambda C(\mathbf{w},\mathbf{v},\mathbf{y}') + (1-\lambda) C(\mathbf{w},\mathbf{v},\mathbf{y}'')$$

By feasibility, the infimum has to be dominated by the left-hand side establishing convexity. Sublinearity is then established by establishing that C is positively linearly homogenous in $\mathbf{y}$.

$$\begin{aligned}
C(\mathbf{w},\mathbf{v},\mu\mathbf{y}) &= \inf_{\mathbf{r},\mathbf{z}} \left\{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \mathbf{r}+\mathbf{z} \geq \mu\mathbf{y}\right\} \\
&= \inf_{\mathbf{r},\mathbf{z}} \left\{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \frac{\mathbf{r}+\mathbf{z}}{\mu} \geq \mathbf{y}\right\} \\
&= \mu \inf_{\mathbf{r},\mathbf{z}} \left\{c\left(\mathbf{w},\frac{\mathbf{z}}{\mu}\right) + p\left(\mathbf{v},\frac{\mathbf{r}}{\mu}\right) : \frac{\mathbf{r}+\mathbf{z}}{\mu} \geq \mathbf{y}\right\} \\
&= \mu C(\mathbf{w},\mathbf{v},\mathbf{y}).
\end{aligned}$$

This establishes 1.

$C(\mathbf{w},\mathbf{v},\mathbf{0}) \leq 0$ follows by the fact that $c(\mathbf{w},\mathbf{0}) = 0$ and $p(\mathbf{v},\mathbf{0}) \leq 0$.

To establish 3,

$$\begin{aligned}
C\left(\mathbf{w},\mathbf{v},\mathbf{y}+\boldsymbol{\delta}\mathbf{A}_j\right) &= \min_{\mathbf{r},\mathbf{z}} \left\{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \mathbf{r}+\mathbf{z} \geq \mathbf{y}+\boldsymbol{\delta}\mathbf{A}_j\right\} \\
&= \min_{\mathbf{r},\mathbf{z}} \left\{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \mathbf{r}-\boldsymbol{\delta}\mathbf{A}_j+\mathbf{z} \geq \mathbf{y}\right\} \\
&= \delta v_j + \min_{\mathbf{r}-\boldsymbol{\delta}\mathbf{A}_j,\mathbf{z}} \left\{c(\mathbf{w},\mathbf{z}) + p\left(\mathbf{v},\mathbf{r}-\boldsymbol{\delta}\mathbf{A}_j\right) : \mathbf{r}-\boldsymbol{\delta}\mathbf{A}_j+\mathbf{z} \geq \mathbf{y}\right\} \\
&= C(\mathbf{w},\mathbf{v},\mathbf{y}) + \delta v_j,
\end{aligned}$$

where the third equality follows from (5).

8 References

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