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Separability of Stochastic Production Decisions from Producer Risk Preferences in the Presence of Financial Markets

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Abstract: This paper presents a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions on the firm's stochastic technology and objective function. The specific focus is on separation results for stochastic technologies, that is, on conditions under which the optimal production decision may be determined without regard to the risk preferences of the firm's owners. Necessary and sufficient conditions for separation, which generalize existing results, are presented. We show, among other results, that separation implies that the linear pricing of assets in the span of the market can be extended to encompass sets of assets outside of the span that are not perfectly replicable.

Key words: stochastic production, financial markets, separation, asset pricing

Separability of Stochastic Production Decisions from Producer Risk Preferences in the Presence of Financial Markets

Separation results, as they are usually understood, refer to conditions under which a firm's production decisions are independent of its risk attitudes. Well-understood situations where separation occurs typically include those where technically feasible production opportunities are replicable in financial markets (Magill and Quinzii, 1995). This phenomenon is usually referred to as 'spanning'.

Analysis of separation has focused on the derivation of 'spanning results'. The best-known such result is that of complete markets. Another well-known spanning result is that, with a single-output non-stochastic technology and stochastic prices, the production choice of an expected-utility maximizer is independent of its risk preferences if there exists an active forward market (Danthine, 1978; Holthausen, 1979; and Anderson and Danthine, 1981, 1983a, 1983b). A generalization of both of these spanning results occurs when a convex but stochastic production set lies completely within the span of existing financial assets (Magill and Quinzii, 1995; Milne, 1995).

The paper gives necessary and sufficient conditions for separation that go beyond these well-understood spanning conditions. To do so, we present a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions about the firm's technology and objective function.

There are at least three reasons why such results are important. First, there is the potential analytical and empirical convenience that arises from being able to ignore a firm's idiosyncratic risk attitudes in studying and empirically modelling its production decisions.

Second, and perhaps more important in the long run, separation has important implications for asset valuation in incomplete markets. There are two dominant approaches to asset pricing: consumption-based pricing and arbitrage pricing. For assets within the span of the market, the two approaches coincide. But in the absence of perfect replicability, arbitrage pricing only places upper and lower bounds on asset prices. These bounds are frequently so imprecise as to be economically irrelevant (Bernardo and Ledoit, 2000;

Cochrane, 2001). On the other hand, consumption-based pricing, while offering exactness for nonreplicable assets, lacks robustness, because results depend on potentially arbitrary assumptions on preferences.[1]

The notions of idiosyncratic risk and real options based on nontradable securities suggests that firms typically need to value nonreplicable assets. Thus, it is important to identify and to understand instances when exact pricing is available even in the absence of perfect replicability. If separation occurs, *sets of nonreplicable assets can be exactly priced using the market pricing kernel* without requiring arbitrary restrictions on preferences stronger than simple monotonicity. Thus, if separation occurs, the market-based pricing functional can be extended beyond the span of the market to encompass linear pricing of some nonreplicable assets. Separation conditions, therefore, provide an alternative to the noisy arbitrage-bound approach and the less robust consumption-based asset pricing approach.

Third, if all firms operating in a sole-proprietorship equilibrium satisfy the conditions for separation, the resulting equilibrium is Pareto optimal. Moreover, since, as we show, *separation does not require that the firm's net consumption choices lie in the span of the market,* the presence of separation extends the range of Paretian equilibria for incomplete markets beyond the well-understood cases of effectively complete markets (LeRoy and Werner, 2000) and linear risk tolerant preferences meeting an extreme form of Gorman's (1953) aggregation criteria (Milne, 1995). It is of particular note to emphasize that separation conditions are defined in terms of restrictions on the firm's technology and not its preferences.

Our main analytical tool is the derivative-cost function, which gives the minimum cost of achieving a state-contingent return vector through a combination of production choices and trade in financial assets. The derivative-cost function is closely related to the asset pricing functionals underlying the arbitrage and asset pricing results of Ross (1976, 1987), Prisman (1986), Clark (1993), and others.

In what follows, we first present our notation and some basic results from convex

[1]The less familiar production-based approach to asset pricing (Cochrane, 1991) can also be criticized on these ground.

analysis. We then specify the firm's stochastic environment, both in terms of production opportunities and its access to financial markets. Next we characterize the derivative-cost function and briefly develop its most relevant properties. After that we derive necessary and sufficient conditions for separation. Then the paper concludes.

1 Notation and Preliminaries

For a convex function[2] $f : \Re^S \to \Re$, its *subdifferential* at $\mathbf{m}$ is the closed, convex set:

$$\partial f(\mathbf{m}) = \left\{ \mathbf{k} \in \Re^S : f(\mathbf{m}) + \mathbf{k}(\mathbf{m}' - \mathbf{m}) \leq f(\mathbf{m}') \text{ for all } \mathbf{m}' \right\}. \quad (1)$$

The elements of $\partial f(\mathbf{m})$ are referred to as *subgradients.* If f is differentiable at $\mathbf{m}$, $\partial f(\mathbf{m})$ is a singleton and corresponds to the usual gradient. Conversely, if $\partial f(\mathbf{m})$ is a singleton, f is differentiable at $\mathbf{m}$.

For f convex, its convex conjugate is denoted

$$f^*(\mathbf{k}) = \sup_{\mathbf{m}} \{\mathbf{km} - f(\mathbf{m})\}.$$

If f is proper and closed,[3] then f^* is also a proper and closed convex function with

$$f(\mathbf{m}) = \sup_{\mathbf{k}} \{\mathbf{km} - f^*(\mathbf{k})\}, \quad (2)$$

and on the relative interior of their domains

$$\mathbf{k} \in \partial f(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial f^*(\mathbf{k}). \quad (3)$$

2 State-Contingent Technologies and the Asset Structure

We model a sole-proprietorship, price-taking firm facing a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain.

[2] These results on convex functions are all drawn directly from Rockafellar (1970).

[3] f is proper if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$, and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$. A proper convex function is closed if it is lower-semicontinuous.

Uncertainty is resolved by 'Nature' making a choice from $\Omega = \{1, 2, ..., S\}$. Each element of Ω is referred to as a state of nature. The only assumption on the firm's preferences is that they are increasing in period 0 consumption and period 1 consumption.

The firm's stochastic production technology is represented by a single-product, state-contingent input correspondence. Let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \Re_+^S$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0. If state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input–output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output in period 1 is z_s.

The continuous input correspondence, $X : \Re_+^S \to \Re_+^N$, maps state-contingent output vectors into input sets that are capable of producing them:

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

We impose the following properties on $X(\mathbf{z})$:

X.1 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.2 $\lambda X(\mathbf{z}) + (1-\lambda)X(\mathbf{z}') \subseteq X(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \qquad 0 \leq \lambda \leq 1$.

X.3 X is continuous.

X.1 says that if an input combination can produce a particular mix of state-contingent outputs then it can produce a smaller mix of state-contingent outputs. X.2 ensures that the graph of the input correspondence,

$$T = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in X(\mathbf{z})\},$$

is convex. Convexity of the graph ensures that the technology exhibits diminishing returns in its usual sense.

Period 0 input prices are denoted by $\mathbf{w} \in \Re_+^N$ and are non-stochastic. Financial markets are frictionless, and the *ex ante* financial security payoffs (measured in the same units as $\mathbf{z}$) are given by the $S \times J$ non-negative matrix $\mathbf{A}$. The vector of state-contingent payoffs on the jth financial asset is denoted $\mathbf{A}_j \in \Re_+^S$, and its price is denoted v_j. The firm's portfolio vector, corresponding to the period 0 purchases of the financial assets, is denoted $\mathbf{h} \in \Re^J$. With little true loss of generality, we assume that $\mathbf{A}$ is of full column rank so that there are no redundant assets. Denote the span of the financial markets by $M \subset \Re^S$,

$$M = \left\{ \mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{h}, \mathbf{h} \in \Re^J \right\}.$$

2.1 Production cost structure

Dual to $X(\mathbf{z})$ is the production cost function, $c : \Re_+^N \times \Re_+^S \rightarrow \Re_+$,

$$c(\mathbf{w}, \mathbf{z}) = \min_{\mathbf{x}}\{\mathbf{w}\mathbf{x} : \mathbf{x} \in X(\mathbf{z})\} \quad \mathbf{w} \in \Re_+^N$$

if there exists an $\mathbf{x} \in X(\mathbf{z})$ and ∞ otherwise. $c(\mathbf{w}, \mathbf{z})$ is equivalent to the multi-product cost function familiar from non-stochastic production theory (Färe 1988). If the input correspondence satisfies properties X, $c(\mathbf{w}, \mathbf{z})$ satisfies (Chambers and Quiggin, 2000): $\mathbf{z}^o \geq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}^o) \geq c(\mathbf{w}, \mathbf{z})$; and $c(\mathbf{w}, \mathbf{z})$ is convex on $\Re_+^S$ and continuous on the interior of the region where it is finite.[4]

Let $\mathbf{q} \in \Re_+^S$ denote a vector of present-value prices (state-claim prices, a stochastic discount factor) that the firm faces. The convex conjugate of c,

$$c^*(\mathbf{w}, \mathbf{q}) = \sup_{\mathbf{z}} \{\mathbf{q}\mathbf{z} - c(\mathbf{w}, \mathbf{z})\},$$

is the present-value profit-function for the present-value prices $\mathbf{q}$. Let $\mathbf{z}' \in \arg\sup \{\mathbf{q}\mathbf{z} - c(\mathbf{w}, \mathbf{z})\}$, then

$$\mathbf{q}\mathbf{z}' - c(\mathbf{w}, \mathbf{z}') \geq \mathbf{q}\mathbf{z} - c(\mathbf{w}, \mathbf{z})$$

for all $\mathbf{z}$, and hence $\mathbf{q} \in \partial c(\mathbf{w}, \mathbf{z}')$. By (3), we then obtain $\mathbf{z}' \in \partial c^*(\mathbf{w}, \mathbf{q})$, which restates Hotelling's lemma in terms of subdifferentials. Denote

$$P(\mathbf{w}) = \{\mathbf{q} : c^*(\mathbf{w}, \mathbf{q}) < \infty\}.$$

[4]Because it is a cost function, c, also satisfies monotonicity and curvature properties in $\mathbf{w}$. We do not use these in what follows, and they are therefore not discussed. Chambers and Quiggin (2000) contains a complete discussion.

2.2 Asset valuation in financial markets

Dual to $\mathbf{A}$ is the valuation functional (for example, Prisman, 1986; and Ross, 1987) for $\mathbf{r}$, $p : \Re^J_+ \times \Re^S_+ \to \Re$ defined by the linear program

$$p(\mathbf{v}, \mathbf{r}) = \min \{\mathbf{vh} : \mathbf{Ah} \geq \mathbf{r}\},$$

if $\{\mathbf{h} : \mathbf{Ah} \geq \mathbf{y}\}$ is nonempty and ∞ otherwise.

p is the price of the cheapest portfolio that dominates $\mathbf{r}$. Its basic properties are well-known. It is sublinear in $\mathbf{r}$ and $p(\mathbf{v}, \mathbf{0}) \leq 0$.

The absence of arbitrage can be defined in terms of $p(\mathbf{v}, \mathbf{r})$ (Prisman, 1986; Ross, 1987). An arbitrage exists if there is either a zero-priced portfolio for which $\mathbf{r} \geq \mathbf{0}$ but $\mathbf{r} \neq \mathbf{0}$, or if there is a negatively priced portfolio for which $\mathbf{r} = \mathbf{0}$ (Ross, 1976; Prisman, 1986; Ross, 1987; Magill and Quinzii, 1995; LeRoy and Werner, 2000). Thus, the absence of arbitrage requires $p(\mathbf{v}, \mathbf{r}) > 0$ for $\mathbf{r} \geq \mathbf{0}$ with $\mathbf{r} \neq \mathbf{0}$ and $p(\mathbf{v}, \mathbf{0}) \geq 0$. By the latter and the basic properties of p, the absence of arbitrage thus requires that $p(\mathbf{v}, \mathbf{0}) = 0$.

Dual to p is the present-value arbitrage profit function defined as the convex conjugate of p

$$p^*(\mathbf{v}, \mathbf{q}) = \sup_{\mathbf{r}} \{\mathbf{qr} - p(\mathbf{v}, \mathbf{r})\}.$$

Because p is sublinear over $\mathbf{r}$, p^* equals either 0 or ∞. By conjugacy, therefore,

$$\begin{aligned} p(\mathbf{v}, \mathbf{0}) &= \sup_{\mathbf{q}} \{-p^*(\mathbf{v}, \mathbf{q})\} \\ &= -\inf_{\mathbf{q}} \{p^*(\mathbf{v}, \mathbf{q})\}. \end{aligned}$$

Because the absence of arbitrage requires that $p(\mathbf{v}, \mathbf{0}) = 0$, by dual conjugacy the absence of arbitrage requires the existence of a set of present-value prices, $\mathcal{N}(\mathbf{v})$,

$$\mathcal{N}(\mathbf{v}) = \{\mathbf{q} : p^*(\mathbf{v}, \mathbf{q}) = 0\}.$$

$\mathcal{N}(\mathbf{v})$ is the *set of no-arbitrage prices.* Alternatively, by (3)

$$\begin{aligned} \mathcal{N}(\mathbf{v}) &= \partial_{\mathbf{r}} p(\mathbf{v}, \mathbf{0}) \\ &= \left\{\mathbf{q} \in \Re^S_+ : \mathbf{qA} = \mathbf{v}\right\}. \end{aligned}$$

For any $\hat{\mathbf{r}} \in M$

$$\begin{aligned} p(\mathbf{v}, \hat{\mathbf{r}}) &= \mathbf{v}\hat{\mathbf{h}} \\ &= \mathbf{v}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\hat{\mathbf{r}} \\ &= \bar{\mathbf{q}}(\mathbf{v})\,\hat{\mathbf{r}}. \end{aligned}$$

Thus,

$$\bar{\mathbf{q}}(\mathbf{v}) \in M.$$

$\bar{\mathbf{q}}(\mathbf{v})$ is referred to as the *pricing kernel* for M.[5] It is the unique orthogonal projection of $\mathcal{N}(\mathbf{v})$ onto M.

In some cases, for example, where market participants agree on a non-uniform probability distribution over the state space S (rational expectations), it may be appropriate to consider a change of basis. For a change of basis represented by a nonsingular $S \times S$ matrix Ω, the change of measure corresponding to $\bar{\mathbf{q}}(\mathbf{v})$ is

$$\bar{\mathbf{q}}(\mathbf{v};\Omega) = \mathbf{v}(\mathbf{A}'\Omega^{-1}\mathbf{A})^{-1}\mathbf{A}'\Omega^{-1}$$

If the pricing kernel $\bar{\mathbf{q}}(\mathbf{v})$ is regarded as an ordinary least-squares estimator, $\bar{\mathbf{q}}(\mathbf{v};\Omega)$ is the corresponding generalized least squares estimator, and $\bar{\mathbf{q}}(\mathbf{v};\Omega)$ satisfies

$$E\bar{\mathbf{q}}(\mathbf{v};\Omega)\,\mathbf{r} = \mathbf{v}\hat{\mathbf{h}},$$

where expectations are taken with respect to the measure associated with Ω.

2.3 The derivative-cost function

Define the *derivative-cost function* $C : \Re^S_+ \to \Re$, by

$$\begin{aligned} C(\mathbf{w}, \mathbf{v}, \mathbf{y}) &= \min_{\mathbf{h},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{z} \geq \mathbf{y}\} \\ &= \min_{\mathbf{r},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + p(\mathbf{v}, \mathbf{r}) : \mathbf{r} + \mathbf{z} \geq \mathbf{y}\}. \end{aligned} \tag{4}$$

[5]It is known variously as the mimicking portfolio, the ideal portfolio, and the ideal discount factor.

C is the firm's internal price of the cheapest asset and production portfolio that dominates $\mathbf{y}$. Chambers and Quiggin (2002) show among other results that C is a nondecreasing, convex function of $\mathbf{y}$ that is continuous on the interior of the region where it is finite. [6]

Given present-value prices, $\mathbf{q} \in \Re_{+}^{S}$ the convex conjugate of C,

$$C^{*}(\mathbf{w}, \mathbf{v}, \mathbf{q}) = \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})\},$$

is the firm's present-value profit function. This present-value profit function is unboundedly large if there exist any arbitrage opportunities in financial markets at $\mathbf{q}$. When there are no arbitrage opportunities, the Fisher separation theorem implies that the firm's present-value profit is given by the maximal present-value profit realized from the production of $\mathbf{z}$ (Milne, 1995; Magill and Quinzii, 1995). In dual terms, for $\mathbf{q} \in \Re_{+}^{S}$, this fact is expressed as

$$\begin{aligned}
C^{*}(\mathbf{w}, \mathbf{v}, \mathbf{q}) &= \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{w}, \mathbf{v}, \mathbf{y})\} \\
&= \sup_{\mathbf{y}} \left\{\mathbf{q}\mathbf{y} - \min_{\mathbf{r},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + p(\mathbf{v}, \mathbf{r}) : \mathbf{r} + \mathbf{z} \geq \mathbf{y}\}\right\} \\
&= \sup_{\mathbf{y},\mathbf{r},\mathbf{z}} \{\mathbf{q}\mathbf{y} - \mathbf{c}(\mathbf{w}, \mathbf{z}) - p(\mathbf{v}, \mathbf{r}) : \mathbf{r} + \mathbf{z} \geq \mathbf{y}\} \\
&= \sup_{\mathbf{r},\mathbf{z}} \{\mathbf{q}(\mathbf{r} + \mathbf{z}) - \mathbf{c}(\mathbf{w}, \mathbf{z}) - p(\mathbf{v}, \mathbf{r})\} \\
&= \begin{cases} \infty & \mathbf{q} \notin \mathcal{N}(\mathbf{v}) \\ c^{*}(\mathbf{w}, \mathbf{q}) & \mathbf{q} \in \mathcal{N}(\mathbf{v}) \end{cases}.
\end{aligned}$$

The conjugacy (2) between $C^{*}(\mathbf{w}, \mathbf{v}, \mathbf{q})$ and $C(\mathbf{w}, \mathbf{v}, \mathbf{y})$ allows us to establish our central structural result on C :

Theorem 1 *If $C(\mathbf{w}, \mathbf{v}, \mathbf{y}) > -\infty$, then $C(\mathbf{w}, \mathbf{v}, \mathbf{y}) = \sup_{\mathbf{q}} \{\mathbf{q}\mathbf{y} - c^{*}(\mathbf{w}, \mathbf{q}) : \mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})\}$.*

If C is proper, it is dually interpreted as the maximal value, taken over the financial and production no-arbitrage present-value prices, that the firm attaches to $\mathbf{y}$ less the present-value of its profit. Or put another way, it is the upper bound that the firm attaches to its portfolio and production cost given that it chooses a total position of $\mathbf{y}$. Given that

[6] C is also a cost function, and it has standard properties of cost functions in the prices $(\mathbf{w}, \mathbf{v})$ which are not relevant to our argument, and therefore we do not discuss.

its production cost represents its internal price for $\mathbf{z}$, the derivative-cost function is the upper bound (maximal buying price) on the firm's valuation of $\mathbf{y}$. The corresponding lower bound is

$$\inf_{\mathbf{q}} \{\mathbf{qy}-c^* (\mathbf{w}, \mathbf{q}) : \mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\}.$$

These bounds are of interest in their own right (Chambers and Quiggin, 2002). Because they are defined over $P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})$ instead of $\mathcal{N}(\mathbf{v})$, they are 'tighter' than the no-arbitrage bounds. Thus, recognizing the firm's feasible production opportunities provides an alternative to the 'mixed-approach' methods of tightening no-arbitrage bounds. Moreover, the production-based bounds can be tightened even further by introducing restrictions on the acceptable volatility of $\mathbf{q}$ as in the mixed approaches of Bernardo and Ledoit (2000) and Cochrane and Sa á-Requejo (2000). In the case of complete markets, $\mathcal{N}(\mathbf{v})$ is a singleton and corresponds to $\mathbf{A}^{-1}\mathbf{v}$. Theorem 1 then implies

$$C(\mathbf{w}, \mathbf{v}, \mathbf{y}) = \left(\mathbf{A}^{-1}\mathbf{v}\right)\mathbf{y}-c^*\left(\mathbf{w}, \mathbf{A}^{-1}\mathbf{v}\right).$$

3 Separation

Loosely speaking, separation conditions ensure that different decisionmakers with different risk preferences, but the same technology, make the same production choices. To make this intuition precise, we begin by defining a notion of separation over a set. We say that *separation applies over a set* $Y \subset \Re^S$ if for all $\mathbf{y} \in Y$,

$$\bar{\mathbf{q}}(\mathbf{v}) \in \partial C(\mathbf{w}, \mathbf{v}, \mathbf{y}). \tag{5}$$

Regardless of the $\mathbf{y} \in Y$ that a firm may choose, if separation applies, Theorem 1 implies that its production decisions are guided by maximization of present-value profit for the state-claim prices $\bar{\mathbf{q}}(\mathbf{v})$. Similarly, any two firms operating at two distinct elements of Y, but with the same production technology, make their resource allocation decisions by maximizing present-value profit for the state-claim prices $\bar{\mathbf{q}}(\mathbf{v})$.

Two points should be noted. Separation is defined here in terms of the pricing kernel $\bar{\mathbf{q}}(\mathbf{v})$. This is done for the sake of simplicity and concreteness, and because it is likely to be the type of separation of most theoretical interest. However, in general, separation can potentially occur at other $\mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$ or for any change of measure associated with the pricing kernel. Second, because the profit-maximizing output for prices $\bar{\mathbf{q}}(\mathbf{v})$ need not be unique, separation over Y does not imply that all producers with $\mathbf{y} \in Y$ must choose the same $\mathbf{z}$. However, by Theorem 1, separation does imply that the firm's present-value profit is given by $c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$. Thus even if two firms operating in Y do choose distinct output bundles, they receive the same present-value returns from their production operations and thus their firms have the same market values.

Any $\mathbf{z}$, which the firm produces, must belong to

$$Z' = \{\mathbf{z}' : p(\mathbf{v}, \mathbf{z}') \geq c(\mathbf{w}, \mathbf{z})\}.$$

For any $\mathbf{z}$ not in this set, the decisionmaker is always better off assembling $\mathbf{z}$ in financial markets. This observation leads us to the crudest kind of separation result. If Z' is empty, then the decisionmaker's production decisions are always independent of his risk attitudes, because all rational decisionmakers would operate at $\mathbf{z} = \mathbf{0}$, regardless of the magnitude of $\mathbf{y}$. This is the case where the technology is entirely redundant in the presence of asset markets.

By theorem 1, if the technology is redundant, $\mathcal{N}(\mathbf{v}) \cap P(\mathbf{w}) = \mathcal{N}(\mathbf{v})$. An example illustrates.

Example 1 *There is a single asset traded, priced at one, whose stochastic payout is given by*

$$\mathbf{A}_1 = (a_1, a_2, ..., a_S) > \mathbf{0}.$$

$\mathcal{N}(\mathbf{v}) = \left\{\mathbf{q} \in \Re^S_+ : \mathbf{q}\mathbf{A}_1 = 1\right\}$. *Consider the stochastic production function with multiplicative productivity shock used in (among others) Diamond (1967), Jermann (1998), and Tallarini (2000), whose input correspondence is given by*

$$X(\mathbf{z}) = \cap_{s=1}^S \{\mathbf{x} : f(\mathbf{x}) a_s \geq z_s\},$$

with f positively linearly homogeneous and concave. The dual cost function is

$$c(\mathbf{w}, \mathbf{z}) = \gamma(\mathbf{w}) \max\left\{\frac{z_1}{a_1}, ..., \frac{z_S}{a_S}\right\}$$

with $\gamma(\mathbf{w})$ *dual to* $f(\mathbf{x})$. *In any optimum,*

$$\frac{z_1}{z_s} = \frac{a_1}{a_s},$$

for all s. *Present value profit is* $\frac{z_1}{a_1}(\mathbf{qA}_1 - \gamma(\mathbf{w}))$. *Thus,*

$$P(\mathbf{w}) = \{\mathbf{q} : \mathbf{qA}_1 \leq \gamma(\mathbf{w})\}.$$

Over $\Re^S_+$, *this technology replicates* $\mathbf{A}_1$. *If* $\gamma(\mathbf{w}) > 1$, *it is redundant in the face of* $\mathbf{A}_1$ *priced at one.*

Denote

$$Z^o = \cup_{\mathbf{x}} \{\mathbf{z} : \mathbf{x} \in X(\mathbf{z})\},$$

$$Z^* = \cup_{\mathbf{q} \in P(\mathbf{w})} \{\mathbf{z} : \mathbf{z} \in \partial c^*(\mathbf{w}, \mathbf{q})\}.$$

Z^o is the set of technically feasible state-contingent outputs and Z^* represents the set of state-contingent outputs that are economically efficient for some present-value price vector that does not lead to infinitely large present-value profit. Thus, it might be thought of as the *efficient set* because it represents the set of state-contingent outputs that could be chosen by a present-value profit maximizer.

If $Z^o \subset M$, then the firm's production choices are guided by the pricing kernel (see, for example, Magill and Quinzii, 1995, p.351). In our terms, separation applies over M. This is often referred to as *spanning* because it implies that the range of feasible technical choices always lies within the span of the market. Hence, pricing must be consistent with pricing in the market. An obvious and important special case is the case of complete markets.

Although significant, spanning is too restrictive to encompass many reasonable market scenarios. Moreover, it describes a situation where the role of production in managing the firm's potential idiosyncratic risk is trivialized. Theorem 1, however, allows us to restrict attention to $Z^* \subset Z^o$ in considering economically rational choices of the state-contingent output vector. We have:

Theorem 2 *If* $Z^* \subset M$, *then in the optimum*

$$\mathbf{z} \in \partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})).$$

Proof If $Z^* \subset M$, there is no loss of generality in rewriting the derivative-cost problem as

$$\min_{\mathbf{h},\mathbf{r}} \{c(\mathbf{w}, \mathbf{Ah}) + p(\mathbf{v}, \mathbf{r}) : \mathbf{Ah} + \mathbf{r} \geq \mathbf{y}\}.$$

The necessary and sufficient first-order conditions for this problem require

$$\begin{aligned} \mathbf{0} &\in [\partial c(\mathbf{w}, \mathbf{z}) - \boldsymbol{\mu}] \mathbf{A}, \\ \boldsymbol{\mu} &\in \partial p(\mathbf{v}, \mathbf{r}), \end{aligned}$$

where the notation $[\partial c(\mathbf{w}, \mathbf{z}) - \boldsymbol{\mu}] \mathbf{A}$ denotes the set obtained by subtracting $\boldsymbol{\mu}$ from each element of $\partial c(\mathbf{w}, \mathbf{z})$ and postmultiplying the resulting elements by $\mathbf{A}$. By Theorem 1, $\boldsymbol{\mu} \subset \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$. Hence, optimal $\partial c(\mathbf{w}, \mathbf{z})$ must contain the orthogonal projection of the no-arbitrage prices onto M, $\bar{\mathbf{q}}(\mathbf{v})$, and in the optimum $\mathbf{z} \in \partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$ by (3).■

For optimal $(\mathbf{q}, \mathbf{z})$, $(\bar{\mathbf{q}}(\mathbf{v}) - \mathbf{q})\mathbf{z}$ measures the present-value of the entrepreneurial risk faced by the firm on $\mathbf{z}$. If $Z^* \subset M$, Theorem 1 implies that the entrepreneurial risk is always zero because $\bar{\mathbf{q}}(\mathbf{v}) - \mathbf{q} \perp M$. Thus, the firm makes its production choices on the basis of $(\bar{\mathbf{q}}(\mathbf{v}), \mathbf{w})$ even though, because of the presence of idiosyncratic risk, its consumption choices may be guided by the present-value vector $\mathbf{q}$.

Requiring $Z^* \subset M$, which we shall refer to as *efficient-set spanning,* does not imply spanning in its more usual sense. We demonstrate with an example.

Example 2 *There is a single marketed asset, with payout*

$$\mathbf{A}_1 = (a_1, a_2, ..., a_S) > 0,$$

and the stochastic production technology is

$$c(\mathbf{w}, \mathbf{z}) = \hat{c}\left(\mathbf{w}, \max\left\{\frac{z_1}{a_1}, ..., \frac{z_S}{a_S}\right\}\right),$$

with $\hat{c}$ strictly increasing and strictly convex in its second argument. This cost function is dual to

$$X(\mathbf{z}) = \left\{\mathbf{x} : g(\mathbf{x}) \geq \max\left\{\frac{z_1}{a_1}, ..., \frac{z_S}{a_S}\right\}\right\},$$

with $\mathbf{g}$ *strictly concave. Clearly,* Z^o *is not contained within* M*. In any optimum,*

$$\frac{z_1}{z_s} = \frac{a_1}{a_s}$$

for all s*. Hence,* $\partial c^* (\mathbf{w}, \mathbf{q}) \subset M$ *for all* $\mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$*, and efficient set spanning applies.*

Example 2 proves:

Theorem 3 *Efficient-set spanning does not imply spanning.*

We now state and prove our main result:

Theorem 4 *Separation applies over a set* Y *if and only if* $Y \subset M + \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$.

Proof $\Rightarrow$Suppose that separation applies over a set $Y \subset \Re^S$. By Theorem 1

$$\begin{aligned} C(\mathbf{w}, \mathbf{v}, \mathbf{y}) &= \bar{\mathbf{q}}(\mathbf{v})\mathbf{y} - c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \bar{\mathbf{q}}(\mathbf{v})[\mathbf{z} + \mathbf{r}] - c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) \end{aligned}$$

for $\mathbf{z} \in \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$ with $\bar{\mathbf{q}}(\mathbf{v})\mathbf{r} = p(\mathbf{v}, \mathbf{r})$. Thus, $\mathbf{y} \in M + \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$.

$\Leftarrow$ Let $\mathbf{y} \in M + \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$. There must exist $\mathbf{r} \in M$ and a $\mathbf{z} \in \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$ such that $\mathbf{y}$ can be achieved at a cost of

$$\bar{\mathbf{q}}(\mathbf{v})\mathbf{r} + c(\mathbf{w}, \mathbf{z}).$$

Thus, $C(\mathbf{w}, \mathbf{v}, \mathbf{y}) \leq \bar{\mathbf{q}}(\mathbf{v})\mathbf{r} + c(\mathbf{w}, \mathbf{z})$. Theorem 1 implies

$$C(\mathbf{w}, \mathbf{v}, \mathbf{y}) \geq \mathbf{q}(\mathbf{r} + \mathbf{z}) - c^* (\mathbf{w}, \mathbf{q})$$

for all $\mathbf{q} \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$. But since, $\bar{\mathbf{q}}(\mathbf{v}) \in \mathcal{N}(\mathbf{v}) \cap P(\mathbf{w})$, this later inequality requires

$$\begin{aligned} C(\mathbf{w}, \mathbf{v}, \mathbf{y}) &\geq \bar{\mathbf{q}}(\mathbf{v})\mathbf{r} + \bar{\mathbf{q}}(\mathbf{v})\mathbf{z} - c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) \\ &= \bar{\mathbf{q}}(\mathbf{v})\mathbf{r} + c(\mathbf{w}, \mathbf{z}), \end{aligned}$$

for $\mathbf{z} \in \partial c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$. Hence,

$$C(\mathbf{w}, \mathbf{v}, \mathbf{y}) = \bar{\mathbf{q}}(\mathbf{v})\mathbf{y} - c^* (\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})). \blacksquare$$

Given a technology and an asset structure, Theorem 4 provides a constructive method for finding sets over which separation occurs. Such sets always exist, even though in many instances, they may degenerately equal M. Within these sets, the firm's valuation of its production choices coincides with the market's valuation even in the absence of perfect replicability. Thus, the zone delineates a set of net consumption choices for which the market can accurately price the corresponding production risk faced by firms despite the fact that neither the output produced or the consumption bundle chosen need be replicable in the market. Ordinarily, the failure of replicability implies that the market can only place lower and upper bounds on $\mathbf{z}$, which correspond to the no-arbitrage bounds given by, respectively

$$\sup\{\mathbf{qz} : \mathbf{q} \in \mathcal{N}(\mathbf{v})\} \geq \bar{\mathbf{q}}(\mathbf{v})\,\mathbf{z} \geq \inf\{\mathbf{qz} : \mathbf{q} \in \mathcal{N}(\mathbf{v})\}.$$

Typically, no-arbitrage bounds for nonreplicable assets are very broad and, therefore, convey relatively little pricing information. The broadness of the bounds is a primary impetus behind the development of 'mixed pricing instruments' such as Bernardo and Ledoit's (2000) gain-loss Cochrane and Sa á-Requejo's (2000) good-deal bounds. Separation allows the no-arbitrage bounds to be narrowed to a single value. When separation occurs, the range of assets that can be accurately priced using the market line increases.

Although we do not address general-equilibrium concerns, it is also apparent that if all firms in an economy have their optimal consumption choices lying in a separating set Y, the resulting equilibrium is Pareto-optimal. By Theorem 1 and Theorem 4, these firms would share a common marginal rate of substitution between state-contingent consumption equalling $\bar{\mathbf{q}}(\mathbf{v})$.

Sets over which separation occurs can be either very large or very small relative to M. In the extremes, it can correspond to M and to the set addition of M and $\Re^S_+$. How large or small the set is depends in large part upon the flexibility of the production technology in efficiently removing idiosyncratic risk. This flexibility depends importantly on the firm's elasticity of transformation between state-contingent outputs along its isocost contours. Two polar examples illustrate.

Example 3 *There exists a single marketed asset,* $\mathbf{A}_1$, *whose price is one. The pricing kernel is thus*

$$\bar{\mathbf{q}}(\mathbf{v}) = \frac{\mathbf{A}_1}{\mathbf{A}_1'\mathbf{A}_1}.$$

The production technology is

$$c(\mathbf{w}, \mathbf{z}) = \hat{c}\left(\mathbf{w}, \max\left\{\frac{z_1}{b_1}, ..., \frac{z_S}{b_S}\right\}\right), \tag{6}$$

where $b_s > 0$ *for all* s *and* $\hat{c}$ *is strictly increasing and strictly convex in its second argument. For this technology,* $\partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$ *is unique and proportional to* $\mathbf{b} = (b_1, ..., b_S)$. *Separation thus occurs over a set of the form*

$$Y = M + z\mathbf{b}.$$

Here $M = \{\mathbf{y} : \mathbf{y} = m\mathbf{A}_1, m \in \Re\}$.

The technology in (6) has isocost contours that exhibit no substitutability between efficient state-contingent outputs. The marginal rate of transformation between state-contingent outputs is zero. Thus, this technology's efficient set has almost no ability to ameliorate the idiosyncratic risk outside of M. In effect, this technology is equivalent to making available another financial asset, which is nonlinearly priced and subject to a prohibition on short selling. While this total lack of substitutability may seem somewhat pathological to economists, it is commonly imposed in production-based analyses of asset pricing and the real-business cycle. For example, Jermann (1998), Tallarini (2000), and numerous real-business cycle models employ a multiplicative productivity shock version of this technology (see also Example 1). Moreover, Chambers and Quiggin (2000) have shown that *all stochastic production function representations manifest a similar form of nonsubstitutability* between state-contingent outputs. Even less restrictive stochastic-production function specifications such as that used in Cochrane's (1991) elegant empirical analysis of production-based asset pricing exhibit this form of nonsubstitutability across state-contingent outputs. The possibility of separation, either theoretically or empirically, for these technologies seems remote unless arbitrarily strong restrictions are placed on the firm's underlying preference structure.

Example 4 *There is a single asset priced at one,* $\mathbf{A}_1 > \mathbf{0}$*.Consider the polar production technology:*

$$c(\mathbf{w}, \mathbf{z}) = \gamma(\mathbf{w}) \frac{\mathbf{A}_1}{\mathbf{A}_1'\mathbf{A}_1}\mathbf{z},$$

which is dual to

$$X(\mathbf{z}) = \frac{\mathbf{A}_1\mathbf{z}}{\mathbf{A}_1'\mathbf{A}_1} X^{\gamma}$$

where X^{γ} *is the input set dual to unit cost function* $\gamma(\mathbf{w})$*. If* $\gamma(\mathbf{w}) = 1$*, then* $\partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) = \Re_+^S$ *so that separation occurs over* $M + \Re_+^S$*.*

Example 4, which is unashamedly extreme, depicts a technology that exhibits perfect substitutability of state-contingent outputs at a marginal rate of transformation defined by the pricing kernel. It corresponds to the case where, subject to short selling restrictions, the producer's virtual price of each Arrow state claim is $\bar{q}_s(\mathbf{v})$. Thus, the production technology effectively completes $\Re_+^S$. Separation always rules regardless of the firm's attitudes towards risk. This technology represents the polar case of what Chambers and Quiggin (2000) have referred to as state-allocable costs corresponding to an elasticity of transformation equalling infinity.

Under the conditions of Theorem 4, any two firms producing in the separating set Y and facing a suitably convex technology will make the same production decisions. Or, more precisely, the firms will use the same pricing rule in making its production decisions. It is *not true*, however, that Theorem 4 gives conditions necessary and sufficient for two firms facing the same technology and the same financial market structure to make the same production choices.

Consider two firms, with the same technology, but whose differing risk preferences lead them to wish to cover $\mathbf{y}^1$ and $\mathbf{y}^0$. Further, suppose that $\mathbf{y}^0 \in M + \partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$ but that $\mathbf{y}^1 \notin M + \partial c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v}))$. Can they agree on production choices? The answer is yes. By Theorem 1

$$C(\mathbf{w}, \mathbf{v}, \mathbf{y}^1) = \mathbf{q}^1\mathbf{y}^1 - c^*(\mathbf{w}, \mathbf{q}^1)$$

and the firm's production choice is governed by

$$\mathbf{z}^1 \in \partial c^*(\mathbf{w}, \mathbf{q}^1),$$

or dually

$$\mathbf{q}^1 \in \partial c\left(\mathbf{w}, \mathbf{z}^1\right).$$

For firm 0 to make the same production choice as firm 1, it is only required, however, that

$$\bar{\mathbf{q}}(\mathbf{v}) \in \partial c\left(\mathbf{w}, \mathbf{z}^1\right).$$

So long as the cost function is not smoothly differentiable in $\mathbf{z}$, this can happen, and as Chambers and Quiggin (2000) have shown, stochastic-production function representations admit such a possibility at all economically efficient points.

4 Conclusion

We have presented necessary and sufficient conditions for separation that generalize the existing sufficient complete-market and spanning conditions. We have shown, among other results, that separation implies that the linear pricing of assets in the span of the market can be extended to encompass sets of assets that are not perfectly replicable.

Our results have been developed in a two-period framework from the perspective of a single sole-proprietorship firm that takes input prices and financial asset prices as given. Thus, they depend upon an existing market structure, and the sets over which separation will apply change as the market structure changes. Modelling these changes requires a general-equilibrium treatment that permits $(\mathbf{v}, \mathbf{w})$ to be determined endogenously. This is done in a two-period dual framework by Chambers and Quiggin (2003), where the connection between these necessary and sufficient conditions for separation and Pareto optimality in incomplete markets is examined in more detail.

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