Risk & Sustainable Management Group

Risk & Uncertainty Program Working Paper: 5/R03

Research supported by an Australian Research Council Federation Fellowship
http://www.arc.gov.au/grant_programs/discovery_federation.htm

Comparative statics for state-contingent technologies

Robert G. Chambers
University of Maryland, College Park

and

John Quiggin
Australian Research Council Federation Fellow, University of Queensland

Schools of Economics and Political Science
University of Queensland
Brisbane, 4072
rsmg@uq.edu.au
http://www.uq.edu.au/economics/rsmg

Robert G. Chambers[1] and John Quiggin[2]

30 November 2003

[1]Professor and Adjunct Professor, respectively, University of Maryland and University of Western Australia

[2]Australian Research Council Senior Research Fellow, University of Queensland

Abstract

The implications of supermodularity for comparative-static analysis in a generalized version of the separable-effort representation of a firm facing stochastic prices and a stochastic technology are. Previous analysis is generalized in two ways. General risk-averse, as opposed to expected-utility, preferences are considered. The stochastic technology is represented by an Arrow-Debreu state-space representation. It is shown that results familiar from the theory of the price taking firm in the absence of risk generalize to the uncertain case.

JEL classification: D21, D81

keywords: supermodularity, risk aversion, production

Comparative Statics for State-Contingent Technologies

The concept of equilibrium, and the comparative static analysis of changes from one equilibrium to another, are central to economics. In the last decade, the concept of supermodularity has revolutionized the study of comparative statics, providing a range of tools to unify and extend results previously obtained by techniques such as the manipulation of first-order conditions. There has been particular interest in application of supermodularity concepts to problems involving uncertainty (Milgrom 1994, Athey 2001) in generalizations of the model introduced by Sandmo (1971).

A particularly interesting application of supermodularity is to the analysis of the productive firm under uncertainty by Milgrom (1994). Milgrom's key result is a meta-theorem showing that results derived for a simple linear case analogous to the two-asset portfolio problem can be extended to a range of more general problems considered in the Sandmovian literature. More recently, Athey (2001) has considered firms with objective functions of the general form $U(\mathbf{x},\theta) = \int u(\mathbf{x},\mathbf{s}) f(\mathbf{s},\theta)\, d\mu(\mathbf{s})$, and provided an elegant treatment of monotone comparative static results relating the optimal action vector to changes in θ.

This note studies the implications of supermodularity for comparative-static analysis in a generalized version of the Newbery-Stiglitz (1979, 1981) separable-effort representation of a firm facing stochastic prices and a stochastic technology. Specifically, we generalize their model in two ways. First, we consider general risk-averse preferences, and second we represent production uncertainty by a general Arrow-Debreu state-contingent technology.

Our approach capitalizes on the observation of Arrow and Debreu that the mathematical representation of price and production uncertainty is equivalent to the mathematical representation of a multiple-output firm and on the further recognition that, in supermodularity terms, the separable-effort model is isomorphic to the objective function for a multiple output profit maximizing firm. These observations suggest that results familiar from the theory of multiple-output firm can be transferred directly to the problem of the firm facing price and production uncertainty. This note, in particular, shows that familiar comparative static results for a firm exhibiting cost complementarities (submodular cost structures) for non-stochastic technologies generalize immediately to risk-averse decisionmakers facing a stochastic environment.

1. Notation

Uncertainty is represented by a state space Ω. So that the exposition can exploit parallels with the standard model of multi-output production, we focus on the case where $\Omega = \{1...S\}$ is discrete. However, with a slightly different mathematical apparatus, the results generalize straightforwardly to the case when Ω is an interval with Lebesgue measure or a more general measurable set. We consider preferences over state-contingent income distributions $\mathbf{y} \in \Re^S$, represented by a total ordering $\preceq$. Under standard assumptions of continuity and monotonicity, a canonical representation of preferences is given by the certainty equivalent

$$e(\mathbf{y}) = \inf\{\mu : \mathbf{y} \preceq \mu\mathbf{1}\}.$$

For any preference function over stochastic outcomes, $W : \Re^S \to \Re$, representing $\preceq$, the certainty equivalent may be defined implicitly by the relationship

$$W(e(\mathbf{y})\mathbf{1}) = W(\mathbf{y})$$

where $\mathbf{1} = (1...1) \in \Re^S$ is the unit vector.

For the case of expected-utility preferences,

$$e(\mathbf{y}) = u^{-1}\left(\sum_{s=1}^{S} \pi_s u(y_s)\right), \tag{1}$$

where u is a von Neumann-Morgenstern utility function and the probability vector π is an element of the unit simplex. In our analysis we do not assume the existence of either a von Neumann-Morgenstern utility function or well-defined subjective probabilities.

Let $Y \subset \Re^k$ be a set ordered by the $\leq$ relation.[1] We denote by $\mathbf{x} \vee \mathbf{y}$ the *join* of $\mathbf{x}$ and $\mathbf{y}$ in Y, where

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, ..., \max\{x_S, y_S\}).$$

Denote by $\mathbf{x} \wedge \mathbf{y}$, the *meet* of $\mathbf{x}$ and $\mathbf{y}$ in Y, where

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, ..., \min\{x_S, y_S\}).$$

[1] The terminology and notation here is borrowed directly from Milgrom and Shannon and Topkis.

We denote by $\sqsubseteq$ the strong set ordering induced by $\leq$ on the power set of $\Re^k$. A function h satisfies the *single crossing property in* $(\mathbf{n}, \mathbf{v})$ if with $\mathbf{v}'' > \mathbf{v}'$ and $\mathbf{n}'' > \mathbf{n}'$

$$h(\mathbf{n}'', \mathbf{v}') \geq h(\mathbf{n}', \mathbf{v}') \Rightarrow h(\mathbf{n}'', \mathbf{v}'') \geq h(\mathbf{n}', \mathbf{v}'')$$

A mapping $f : \Re^S \to \Re$ is *supermodular* if for all $\mathbf{y}, \mathbf{y}'$

$$f(\mathbf{y} \vee \mathbf{y}') + f(\mathbf{y} \wedge \mathbf{y}') \geq f(\mathbf{y}) + f(\mathbf{y}').$$

If $-f$ is supermodular then f is *submodular.*

2. The Production Structure

The production technology is modelled by a continuous input correspondence, $X : \Re_+^S \to \Re_+^N$, which maps vectors of outputs, $\mathbf{z}$, into inputs capable of producing them

$$X(\mathbf{z}) = \left\{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\right\} \quad \mathbf{z} \in \Re_+^S.$$

When production is non-stochastic, the interpretation of X is as an input correspondence for a multiple-output firm. When production is stochastic, X represents an input correspondence for a vector of state-contingent outputs. In the latter case, the scalar $z_s \in \Re_+$ denotes the *ex post* or realized output in state s. In addition to continuity, we impose the following axioms on the input correspondence:[2]

X.1 $X(0_S) = \Re_+^N$, and $\mathbf{0}_N \notin X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}_S$ and $\mathbf{z} \neq \mathbf{0}_S$.

X.2 $\mathbf{z}' \geq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.3 $X(\mathbf{z}) = X(\mathbf{z}) + \Re_+^N$, $\mathbf{z} \in \Re_+^S$.

X.4 $X(\mathbf{z})$ is a convex set, $\mathbf{z} \in \Re_+^S$.

As is shown by Chambers and Quiggin (2002), this representation is sufficiently general to encompass not only problems of production under uncertainty but also problems of portfolio choice, with and without taxes and frictions. For the latter case, the vector $\mathbf{x}$ represents purchases of N assets at prices $\mathbf{w}$, and $\mathbf{z}$ is the associated payoff, net of any transactions costs.

[2] Apart from X. 2, these properties are discussed in detail for nonstochastic technologies in Färe (1988) and for state-contingent technologies in Chambers and Quiggin (2000).

Given input prices $\mathbf{w} \in \Re^N_{++}$, we may define the cost function

$$c(\mathbf{w}, \mathbf{z}) = \min\{\mathbf{wx} : \mathbf{x} \in X(\mathbf{z})\}$$

if $X(\mathbf{z})$ is nonempty and ∞ otherwise. By standard results (Färe, 1988, Chambers and Quiggin, 2000), c satisfies

C.1. $c(\mathbf{w}, \mathbf{z})$ is positively linearly homogeneous, non-decreasing, concave, and continuous in $\mathbf{w}$;

C.2. $c(\mathbf{w}, \mathbf{z}) \geq 0$, $c(\mathbf{w}, 0_S) = 0$, and $c(\mathbf{w}, \mathbf{z}) > 0$ for $\mathbf{z} \geq 0_S, \mathbf{z} \neq 0_S$;

C.3. $c(\mathbf{w}, \mathbf{z})$ is nondecreasing and continuous on $\Re^S_{++}$.

Let

$$\mathbf{x}(\mathbf{w}, \mathbf{z}) = \arg\min\{\mathbf{wx} : \mathbf{x} \in X(\mathbf{z})\}.$$

For simplicty, we assume that $\mathbf{x}(\mathbf{w}, \mathbf{z})$ is unique, and thus by Shephard's Lemma $\mathbf{x}(\mathbf{w}, \mathbf{z})$ corresponds to the gradient of the cost function in input prices. Letting ∇ denote the gradient with respect to subscripted vector, the cost function thus satisfies (McFadden, 1978; Färe, 1988)

$$\mathbf{x}(\mathbf{w}, \mathbf{z}) = \nabla_{\mathbf{w}} c(\mathbf{w}, \mathbf{z}). \tag{2}$$

And by standard duality theorems (McFadden, 1978; Färe 1988):

$$X(\mathbf{z}) = \cap_{\mathbf{w} > \mathbf{0}} \{\mathbf{x} : \mathbf{wx} \geq c(\mathbf{w}, \mathbf{z})\}.$$

3. Monotone Comparative Statics under Certainty

For the moment, let us accept the interpretation of $c(\mathbf{w}, \mathbf{z})$ as a multiple-output cost functions for a firm facing competitively determined input prices $\mathbf{w}$ and a competitively determined vector of output prices, $\mathbf{p} \in \Re^S_+$. Our interest here is not in recapitulating the entire received theory of comparative statics for such a firm but in providing a brief review of existing strong monotone comparative static results.[3]

[3]These results, themselves, are well-known, but were originally derived by more direct methods.

We first consider conditions required for all inputs to be *nonregressive* in the sense that:

$$\mathbf{z}' \geq \mathbf{z} \Rightarrow \mathbf{x}(\mathbf{w}, \mathbf{z}') \geq \mathbf{x}(\mathbf{w}, \mathbf{z}),$$

By (2), the requirement that all inputs be nonregressive implies for $\mathbf{w}' > \mathbf{w}$ and $\mathbf{z}' > \mathbf{z}$ that

$$c(\mathbf{w}', \mathbf{z}') - c(\mathbf{w}', \mathbf{z}) \geq c(\mathbf{w}, \mathbf{z}') - c(\mathbf{w}, \mathbf{z}). \tag{3}$$

and hence the cost structure must satisfy increasing differences in $(-\mathbf{w}, \mathbf{z})$. Using the duality mapping, it is evident that (3) is always satisfied if X.2 is strengthened to $\mathbf{z}' \geq \mathbf{z} \Rightarrow X(\mathbf{z}) \sqsubseteq X(\mathbf{z}')$. Notice, in particular, that this comparative static result applies regardless of whether $c(\mathbf{w}, \mathbf{z})$ is interpreted as a multiple-output cost function or whether it is interpreted as the cost function for a state-contingent production technology.

Turning to the profit maximization problem, the firm now seeks to maximize $\mathbf{pz} - c(\mathbf{w}, \mathbf{z})$. Denote

$$\mathbf{z}(\mathbf{w}, \mathbf{p}) = \arg\max \{\mathbf{pz} - c(\mathbf{w}, \mathbf{z})\},$$

and to avoid unnecessary ambiguities, assume that the solution, when one exists, is unique. Our emphasis here is on strong monotone comparative static results in terms of $\mathbf{p}$ and $\mathbf{w}$ for $\mathbf{z}(\mathbf{w}, \mathbf{p})$ and $\mathbf{x}(\mathbf{w}, \mathbf{p})$.

We first consider $\mathbf{z}(\mathbf{w}, \mathbf{p})$. Theorem 4 of Milgrom and Shannon (1994) shows that $\mathbf{z}(\mathbf{w}, \mathbf{p})$ is increasing in $\mathbf{p}$ and decreasing in $\mathbf{w}$ if and only if $\mathbf{pz} - c(\mathbf{w}, \mathbf{z})$ is quasi-supermodular in $\mathbf{z}$ and satisfies the single crossing property in $(\mathbf{z}, \mathbf{p})$ and in $(\mathbf{z}, -\mathbf{w})$. Theorem 2.6.5 of Topkis establishes that the objective function is quasi-supermodular in $\mathbf{z}$ if and only if $c(\mathbf{w}, \mathbf{z})$ is submodular in $\mathbf{z}$. Submodularity of the cost function requires that all outputs be complementary in production.

Under properties C, $\mathbf{pz} - c(\mathbf{w}, \mathbf{z})$ always satisfies the single crossing property in $(\mathbf{z}, \mathbf{p})$. Our first lemma gives a necessary and sufficient condition for $\mathbf{pz} - c(\mathbf{w}, \mathbf{z})$ to satisfy the single crossing property in $(\mathbf{z}, -\mathbf{w})$ for all $\mathbf{p}$. (All proofs are in an Appendix.)

Lemma 1. *$\mathbf{pz} - c(\mathbf{w}, \mathbf{z})$ satisfies the single crossing property in $(-\mathbf{w}, \mathbf{z})$ for all $\mathbf{p}$ if and only if $c(\mathbf{w}, \mathbf{z})$ satisfies (3).*

Using Lemma 1 and Theorem 4 of Milgrom and Shannon establishes the following well-known result for the multiple-output firm:

Theorem 2. $\mathbf{p}' \geq \mathbf{p} \Rightarrow \mathbf{z}(\mathbf{w}, \mathbf{p}') \geq \mathbf{z}(\mathbf{w}, \mathbf{p})$ *and* $\mathbf{w}' \geq \mathbf{w} \Rightarrow \mathbf{z}(\mathbf{w}', \mathbf{p}) \leq \mathbf{z}(\mathbf{w}, \mathbf{p})$ *if and only if* $c(\mathbf{w}, \mathbf{z})$ *is submodular in* $\mathbf{z}$ *and satisfies (3).*

Upon recognizing that profit maximizing input demands are given by

$$\mathbf{x}(\mathbf{w}, \mathbf{p}) = \mathbf{x}(\mathbf{w}, \mathbf{z}(\mathbf{w}, \mathbf{p})),$$

we have:

Corollary 3. *If* $c(\mathbf{w}, \mathbf{z})$ *is submodular in* $\mathbf{z}$ *and satisfies (3),* $\mathbf{p}' \geq \mathbf{p} \Rightarrow \mathbf{x}(\mathbf{w}, \mathbf{p}') \geq \mathbf{x}(\mathbf{w}, \mathbf{p})$ *and* $\mathbf{w}' \geq \mathbf{w} \Rightarrow \mathbf{x}(\mathbf{w}', \mathbf{p}) \leq \mathbf{x}(\mathbf{w}, \mathbf{p})$.

4. The State-Contingent Firm

The preceding results are standard results from the theory of the multiproduct firm dressed in the guise of monotone comparative statics. Our goal in this section is to show that the supermodular structure of the problem under conditions of certainty is directly inherited by a whole class of production models familiar from the literature on production under uncertainty. This formal similarity allows us to transfer these standard results directly to the problem of producers under conditions of price and production uncertainty. Here $\mathbf{z}$ now assumes its interpretation as a vector of state-contingent outputs, and $\mathbf{p}$ is the corresponding vector of state-contingent prices. The producer's objective function is of the form

$$h(\mathbf{z}, \mathbf{p}, \mathbf{w}) = e(p_1 z_1, p_2 z_2, ..., p_S z_S) - c(\mathbf{w}, \mathbf{z}).$$

We, hereafter, maintain the assumption that $c(\mathbf{w}, \mathbf{z})$ is submodular in $\mathbf{z}$. For simplicity, we also continue to assume that the solution to this problem, which we continue to denote by $\mathbf{z}(\mathbf{w}, \mathbf{p})$ is unique. By Theorem 4 of Milgrom and Shannon (1994), we know that the solution to this maximization problem will be monotonic in the parameters $(\mathbf{p}, \mathbf{w})$ if and only if the objective function is quasi-supermodular and satisfies the respective single crossing properties. If this condition is met, then it follows immediately that Theorem 2 applies directly to this class of models. We first slightly generalize Theorem 2.8.5 of Topkis to cover the problem at hand.

Lemma 4. *$h(\mathbf{z}, \mathbf{p}, \mathbf{w})$ is quasi-supermodular in $\mathbf{z}$ for the entire class of submodular cost structures if and only if e is supermodular in $\mathbf{z}$.*

We also have the following extension of Lemma 1:

Lemma 5. *$h(\mathbf{z}, \mathbf{p}, \mathbf{w})$ satisfies the single crossing property in $(\mathbf{z}, \mathbf{p})$ for all cost structures if and only if $e(p_1 z_1, p_2 z_2, ..., p_S z_S)$ satisfies increasing differences in $(\mathbf{z}, \mathbf{p})$. $h(\mathbf{z}, \mathbf{p}, \mathbf{w})$ satisfies the single crossing property in $(\mathbf{z}, -\mathbf{w})$ for all monotonic preferences if and only if c satisfies (3)*

On the basis of these results, we have our first main theorem:

Theorem 6. *If e is supermodular in $\mathbf{p}$ and $\mathbf{z}$, satisfies increasing differences in $(\mathbf{z}, \mathbf{p})$, and c is submodular in $\mathbf{z}$*

$$\mathbf{p}' \geq \mathbf{p} \Rightarrow \mathbf{z}(\mathbf{w}, \mathbf{p}') \geq \mathbf{z}(\mathbf{w}, \mathbf{p}).$$

If e is supermodular, and c is submodular in $\mathbf{z}$ and satisfies (3), then

$$\mathbf{w}' \geq \mathbf{w} \Rightarrow \mathbf{z}(\mathbf{w}', \mathbf{p}) \leq \mathbf{z}(\mathbf{w}, \mathbf{p}).$$

An immediate corollary of Theorem 6 is available for input demands.

Corollary 7. *If e is supermodular in $\mathbf{p}$ and $\mathbf{z}$, satisfies increasing differences in $(\mathbf{z}, \mathbf{p})$,and c is submodular in $\mathbf{z}$*

$$\mathbf{p}' \geq \mathbf{p} \Rightarrow \mathbf{x}(\mathbf{w}, \mathbf{p}') \geq \mathbf{x}(\mathbf{w}, \mathbf{p}).$$

If e is supermodular in $\mathbf{p}$ and $\mathbf{z}$, c is submodular in $\mathbf{z}$ and c is submodular in $\mathbf{z}$ and satisfies increasing differences in $(\mathbf{z}, -\mathbf{w})$, then

$$\mathbf{w}' \geq \mathbf{w} \Rightarrow \mathbf{x}(\mathbf{w}', \mathbf{p}) \leq \mathbf{x}(\mathbf{w}, \mathbf{p}).$$

This analysis can be used to generalize a wide range of existing results. Let e be the certainty equivalent for smoothly differentiable expected utility preferences of the form (1). Then we observe that e is supermodular for all p and z if and only if $0 \leq r < 1$ where r is the coefficient of relative risk aversion. This can be seen as follows. For this case,

$$\frac{\partial^2 u(p_s z_s)}{\partial p_s \partial z_s} = u''(p_s z_s) + u'(p_s z_s) p_s z_s$$

which is positive if and only

$$\begin{aligned} r &= -\frac{u''(p_s z_s)}{u'(p_s z_s) p_s z_s} \\ &< 1 \end{aligned}$$

If this condition holds for all p, z, the result follows from Lemma 2.6.4 of Topkis, provided u^{-1} is convex, which will be true if and only if $r \geq 0$. Thus our results generalize a wide range of existing comparative static results derived under the assumption of expected utility preferences (Newbery and Stiglitz, 1979, 1981).

More generally, consider any objection function of the form

$$W(p_1 z_1, p_2 z_2, ..., p_S z_S) - c(\mathbf{w}, \mathbf{z})$$

where W is twice differentiable and c is submodular in $\mathbf{z}$. For our next result, we need some new notation. Define $g : \Re^S_+ \to \Re^S$ by and

$$W(p_1 z_1, p_2 z_2, ..., p_S z_S) = g(\log p_1 + \log z_1, \log p_2 + \log z_2 +, .., \log p_S + \log z_S),$$

and let $\zeta_s = \log p_s + \log z_s$.The next theorem then follows by differentiation

Theorem 8. *Any twice differentiable* $W : \Re^S_+ \to \Re^S$ *is supermodular in* $\mathbf{p}, \mathbf{z}$ *if and only if*

$$\frac{\partial^2 g}{\partial \zeta_s \partial \zeta_t} \geq 0 \quad \forall s, t \tag{4}$$

Theorem 8 provides an alternative approach to generalization of the results of Newbery and Stiglitz. Consider the logarithmic utility function, with coefficient of relative risk aversion equal to 1. If W corresponds to expected log utility, g is the arithmetic mean which is a valuation. Hence, for any W more risk averse than log utility, g has the properties required and W is supermodular in $\mathbf{p}$ and $\mathbf{z}$.

5. Concluding Comments

The concept of supermodularity permits both simplification and generalization of a wide range of comparative-static results. Using this approach we have generalized the analysis of Newbery and Stiglitz to encompass general risk-averse preferences in place of the empirically questionable assumption of expected-utility maximization. More significantly, the analysis applies to general state-contingent production technologies with arbitrary numbers of inputs and state-contingent outputs, in place of the stochastic production function approach used by Newbery and Stiglitz. We have shown that comparative-static results familiar form nonstochastic firm theory generalize to this case.

6. Reference List

Athey, S. "Monotone Comparative Statics Under Uncertainty." Quarterly Journal of Economics (forthcoming).

Chambers, R. G., and J. Quiggin. Uncertainty, Production, Choice, and Agency: The State-Contingent Approach. New York: Cambridge University Press, 2000.

Färe, R. Fundamentals of Production Theory. Berlin: Springer-Verlag, 1988.

McFadden, D. "Cost, Revenue, and Profit Functions." Production Economics: A Dual Approach to Theory and Applications. eds. M. Fuss, and D.McFaddenAmsterdam: North-Holland, 1978.

Milgrom, P., and C. Shannon. "Monotone Comparative Statics." Econometrica 62 (1994): 157-80.

Milgrom, Paul. "Comparing Optima: Do Simplifying Assumptions Affect Conclusions." Journal of Political Economy, no. 3 (1994): 607-15.

Newbery, D. M. G., and J. Stiglitz. The Theory of Commodity Price Stabilisation–A Study in the Economics of Risk. Oxford: Oxford University Press, 1981.

Sandmo, A. "On the Theory of the Competitive Firm Under Price Uncertainty." American Economic Review 61 (1971): 65-73.

Topkis, D. M. Supermodularity and Complementarity. Princeton: Princeton University Press, 1998.

7. Appendix: Proofs

Proof of Lemma 1: Sufficiency is obvious. For $-\mathbf{w}' > -\mathbf{w}$, $\mathbf{z}' > \mathbf{z}$ the single crossing property requires that

$$\mathbf{p}(\mathbf{z}' - \mathbf{z}) \geq c(\mathbf{w}, \mathbf{z}') - c(\mathbf{w}, \mathbf{z}) \Rightarrow \mathbf{p}(\mathbf{z}' - \mathbf{z}) \geq c(\mathbf{w}', \mathbf{z}') - c(\mathbf{w}', \mathbf{z})$$

Because $\mathbf{p}$ can take any value in $\Re_+^S$, this requires

$$c(\mathbf{w}, \mathbf{z}') - c(\mathbf{w}, \mathbf{z}) \geq c(\mathbf{w}', \mathbf{z}') - c(\mathbf{w}', \mathbf{z}).$$

Proof of Lemma 4: Supermodularity of the cost structure requires

$$c(\mathbf{w}, \mathbf{z}) - c(\mathbf{w}, \mathbf{z} \wedge \mathbf{z}') \geq c(\mathbf{w}, \mathbf{z} \vee \mathbf{z}') - c(\mathbf{w}, \mathbf{z}').$$

Setting $\mathbf{p} = \mathbf{1}$, quasi-supermodularity requires that

$$e(\mathbf{z}) - e(\mathbf{z} \wedge \mathbf{z}') \geq c(\mathbf{w}, \mathbf{z}) - c(\mathbf{w}, \mathbf{z} \wedge \mathbf{z}') \Rightarrow e(\mathbf{z} \vee \mathbf{z}') - e(\mathbf{z}') \geq c(\mathbf{w}, \mathbf{z} \vee \mathbf{z}') - c(\mathbf{w}, \mathbf{z}').$$

Thus,

$$\begin{aligned} e(\mathbf{z}) - e(\mathbf{z} \wedge \mathbf{z}') &\geq c(\mathbf{w}, \mathbf{z}) - c(\mathbf{w}, \mathbf{z} \wedge \mathbf{z}') \\ &\geq c(\mathbf{w}, \mathbf{z} \vee \mathbf{z}') - c(\mathbf{w}, \mathbf{z}') \end{aligned}$$

implies

$$e(\mathbf{z} \vee \mathbf{z}') - e(\mathbf{z}') \geq c(\mathbf{w}, \mathbf{z} \vee \mathbf{z}') - c(\mathbf{w}, \mathbf{z}')$$

for the entire class of submodular functions. Hence,

$$e(\mathbf{z} \vee \mathbf{z}') - e(\mathbf{z}') \geq e(\mathbf{z}) - e(\mathbf{z} \wedge \mathbf{z}').$$

Sufficiency is obvious.

Proof of Lemma 5: The single crossing property requires for $\mathbf{p}' > \mathbf{p}$ and $\mathbf{z}' > \mathbf{z}$ that

$$e(\mathbf{p}\mathbf{z}') - e(\mathbf{p}\mathbf{z}) \geq c(\mathbf{w}, \mathbf{z}') - c(\mathbf{w}, \mathbf{z})$$

implies

$$e(\mathbf{p}'\mathbf{z}') - e(\mathbf{p}'\mathbf{z}) \geq c(\mathbf{w}, \mathbf{z}') - c(\mathbf{w}, \mathbf{z}).$$

which is true for all monotonic cost structures only if

$$e(\mathbf{p}'\mathbf{z}') - e(\mathbf{p}'\mathbf{z}) \geq e(\mathbf{p}\mathbf{z}') - e(\mathbf{p}\mathbf{z}).$$

The converse is trivial.