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The Financial Market Consequences of Growing Awareness: The Case of Implied Volatility Skew

AUTHOR:

Hammad Siddiqi

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Schools of Economics and Political Science

The University of Queensland

St Lucia

Brisbane

Australia 4072

Web: www.uq.edu.au

The Financial Market Consequences of Growing Awareness: The Case of Implied Volatility Skew

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Hammad Siddiqi

University of Queensland

h.siddiqi@uq.edu.au

The belief that the essence of the Black Scholes model is correct implies that one is unaware that *a delta-hedged portfolio is risky*, while believing that the proposition, *a delta-hedged portfolio is risk-free,* is true. Such *partial awareness* is equivalent to *restricted awareness* in which one is unaware of the states in which *a delta-hedged portfolio is risky*. In the continuous limit, two types of restricted awareness are distinguished. 1) *Strongly restricted awareness* in which one is unaware of the type of the true stochastic process. 2) *Weakly restricted awareness*, in which one is aware of the type of the true stochastic process, but is unaware of the true parameter values. We apply the generalized principle of no-arbitrage (analogy making) to derive alternatives to the Black Scholes model in each case. If the Black Scholes model represents *strongly restricted awareness*, then the alternative formula is a generalization of Merton's jump diffusion formula. If the Black Scholes formula represents *weakly restricted awareness*, then the alternative formula, first derived in Siddiqi(2013), is a generalization of the Black Scholes formula. Both alternatives generate implied volatility skew. Hence, the sudden appearance of the skew after the crash of 1987 can be understood as the consequence of growing awareness, as investors realized that a delta-hedged portfolio is risky after suffering huge losses in their portfolio-insurance delta-hedges. The different implications of *strongly restricted awareness* vs. *weakly restricted awareness* for option pricing are discussed.

JEL Classifications: G13; G12

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The Financial Market Consequences of Growing Awareness: The Case of Implied Volatility Skew

On Monday October 19 1987, stock markets in the US (on Tuesday October 20 , in a variety of other markets worldwide), along with the corresponding futures and options markets, crashed; with the S&P 500 index falling more than 20%. To date, in percentage terms, this is the largest ever one-day drop in the value of the index. The crash of 1987 is considered one of the most significant events in the history of financial markets due to the severity and swiftness of market declines worldwide. In the aftermath of the crash, a permanent change in options market occurred; implied volatility skew started appearing in options markets worldwide. Before the crash, implied volatility when plotted against strike/spot is almost a straight line, consistent with the Black Scholes model, see Rubinstein (1994). After the crash, for index options, implied volatility starts falling monotonically as strike/spot rises. That is, the implied volatility skew appeared. What caused this sudden and permanent appearance of the skew? As noted in Jackwerth (2000), it is difficult to attribute this change in behavior of option prices entirely to the knowledge that highly liquid financial markets can crash spectacularly. Such an attribution requires that investors expect a repeat of the 1987 crash at least once every four years, even when a repeat once every eight years seems too pessimistic. Perhaps, the crash not only imparted knowledge that risks are greater than previously thought, it also caused a change in the mental processes that investors use to value options.

Before the crash, a popular market practice was to engage in a strategy known as "portfolio insurance".[1] The strategy involved creating a "synthetic put" to protect equity portfolios by creating a floor below which the value must not fall. However, the creation of a "synthetic put" requires a key assumption dating back to the celebrated derivation of the Black Scholes option pricing formula (Black and Scholes (1973) and Merton (1973)). The key assumption is that an option's payoff can be replicated by a portfolio consisting only of the underlying and a risk free asset and all one needs to

[1] According to Mackenzie (2004), an incomplete list of key players implementing portfolio insurance strategies for large institutional investors during the period leading to the crash of 1987 includes: Leland O'Brien Rubinstein Associates, Aetna Life and Casualty, Putnam Adversary Co., Chase Investors Mgmt., JP Morgan Investment Mgmt., Wells Fargo Investment Advisors, and Bankers Trust Co. See Mackenzie (2004) for details.

do is shuffle money between the two assets in a specified way as the price of the underlying changes. This assumption known as "dynamic replication" is equivalent to saying that "a delta-hedged portfolio is risk-free". The idea of "dynamic replication" or equivalently the idea that "a delta hedged portfolio is risk-free" was converted into a product popularly known as "portfolio insurance", by Leland O'Brien Rubinstein Associates in the early 1980s. The product was replicated in various forms by many other players. "Portfolio insurance" was so popular by the time of the crash, that the Brady Commission report (1988) lists it as one of the factors causing the crash. For further details regarding portfolio insurance and its popularity before the crash, see Mackenzie (2004).

In this article, we argue that before the crash, investors were unaware of the proposition that "a delta-hedged portfolio is risky". That is, they implicitly believed in the proposition that "a delta-hedged portfolio is risk-free". The crash caused "portfolio insurance delta-hedges" to fail spectacularly. The resulting visceral shock drove home the lesson that "a delta-hedged portfolio is risky", thus, increasing investor awareness.

Before the crash, the belief that a delta-hedged portfolio is risk-free led to options being priced based on no-arbitrage considerations. Principle of no arbitrage says that *assets with identical state-wise payoffs should have the same price*, or equivalently, *assets with identical state-wise payoffs should have identical state-wise returns.* So, a delta-hedged portfolio, if it has the same state-wise payoffs as a risk-free asset, should offer the same state-wise returns as the risk-free asset. What if a delta-hedged portfolio does not have state-wise payoffs that are identical to a risk-free asset or any other asset for the matter? Experimental evidence suggests that when people cannot apply the principle of no-arbitrage to value options because they cannot find another asset with identical state-wise payoffs, they rely on a weaker version of the principle, which can be termed the generalized principle of no-arbitrage or analogy making. See Siddiqi (2012) and Siddiqi (2011). The generalized principle of no-arbitrage or analogy making says, *assets with similar state-wise payoffs should have the same state-wise returns on average*, or equivalently, *assets with similar state-wise payoffs should have the same expected return.* The cognitive foundations of this experimentally observed rule are provided by the notion of mental accounting (Thaler (1980), Thaler (1999), and discussion in Rockenbach (2004)), and categorization theories of cognitive science (Henderson and Peterson (1992)). See Siddiqi (2013) for details. The prices determined by the generalized principle of no-arbitrage are arbitrage-free if an equivalent martingale measure exists. Existence of a risk neutral measure or an equivalent martingale measure is both necessary and sufficient for prices to be arbitrage-free. See Harrison and Kreps (1979). We show that the model developed in this article, permits an equivalent martingale measure, hence prices are arbitrage-free.

A call option is widely believed to be a surrogate for the underlying stock as it pays more when the stock pays more and it pays less when the stock pays less.[2] We follow Siddiqi (2013) in taking the similarity between a call option and its underlying as given and apply the generalized principle of no-arbitrage or analogy making to value options.

Li (2008) uses the term *partial awareness* to describe a situation in which one is unaware of a proposition but not of its negation, which is implicitly assumed to be true. So, in Li (2008)'s terminology, investors had *partial awareness* before the crash as they were unaware of the proposition "a delta-hedged portfolio is risky". They implicitly assumed that the proposition "a delta-hedged portfolio is riskless" is true. Even though, it seems natural to characterize states of nature in terms of propositions, it is often useful to refer to the state space directly. Quiggin (2013) points out that there is a mapping between the characterizations in terms of propositions (syntactic representation of unawareness) to the more usual semantic interpretation in which one describes the state space directly. In our case, the semantic interpretation is that, before the crash, investors were unaware of the states in which the delta-hedged portfolio is risky. Such a semantic concept of unawareness is called *restricted awareness*. Grant and Quiggin (2013), and Halpern and Rego (2008) extend the notion of *restricted awareness* to include sub-game perfect and sequential eqilibria in interactive settings.

Quiggin (?)(note to self: ask for reference) proposes an extension of the notion of unawareness to stochastic processes and defines *restricted awareness* as a situation in which one is unaware of at least one state in the discrete stochastic process. For example, let's say the true process is trinomial; however, one is only aware of two states. A person with such *restricted awareness* may create a delta-hedged portfolio which would be risk-free in the two states he is aware of, but if the third state, which he is unaware of, is realized, the portfolio will lose value. That is, he would falsely

[2] As illustrative examples of professional traders considering a call option to be a surrogate of the underlying, see the following posts:
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/,
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

believe that the delta-hedged portfolio is risk-free, whereas in reality, the delta-hedged portfolio is risky. Note, if the third state occurs with a small enough frequency, then one can stay oblivious of it for a considerable time period.

The notion of *restricted awareness* when applied to stochastic processes needs a finer classification in the continuous limit. In the continuous limit, two broad possibilities arise: 1) *Strongly restricted awareness:* One is unaware of the type of the true stochastic process. As one example, suppose the true process is jump-diffusion and one incorrectly thinks that it is ordinary diffusion. 2) *Weakly restricted awareness:* One is aware of the type of the true stochastic process; however, one is unaware of the true parameter values.

Throughout this article, we assume that if one is unaware, then he is unaware that he is unaware. From this point forward, we suppress explicitly mentioning this assumption, for clarity. Section 2 sets up the basic model in discrete time to bring out the economic intuition of *partial* or *restricted awareness* in the context of option pricing. In the continuous limit, if the Black Scholes model represents *strongly restricted awareness*, then the corresponding analogy based formula (assuming full-awareness) is derived in section 2. The formula can be considered a generalization of Merton's jump diffusion formula (see Merton (1976)). If the Black Scholes model represents *weakly restricted awareness,* then the corresponding analogy based formula (assuming full-awareness) is derived in section 3. That formula, first derived in Siddiqi (2013), can be considered a generalization of the Black Scholes model. The implied volatility skew is generated in both cases; hence, the sudden appearance of the skew after the crash of 1987 can be understood as the consequence of growing awareness induced by the crash. Section 4 compares *strongly restricted awareness* with *weakly restricted awareness* and shows that the implied volatility skew as well as the implied volatility smile is generated in the first case, whereas, in the second case, only the skew is explained. Hence, it is argued, that for options on individual stocks, assuming *strongly restricted awareness* is better, whereas for index options, the *weaker* form seems like a natural choice. Section 5 concludes with suggestions for future research.

2. A Discrete-Time Model of Growing Awareness

In this section, we explore the implications of growing awareness in discrete time. This brings out the economic intuition and clarifies the valuation of options by providing a comparison of analogy making and no-arbitrage pricing. The basic set-up of the model is the same as in Amin (1993), which is a generalization of Cox, Ross, and Rubinstein (1979) binomial model (CRR model).

Assume that trade occurs only on discrete dates indexed by 0, 1, 2, 3, 4, ………T. Initially, there are only two assets. One is a riskless bond that pays $\dot{r}$ every period meaning that if B dollars are invested at time i, the payoff at time i+1 is $\dot{r}B$. The second asset is a risky stock. As in Amin (1993), we assume that the stock price at date i can only take values from an exogenously specified set given by $S_j(i)\ where\ j \in \{-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots\dots, \infty\}$. The variable j is an index for state and the variable i is an index for time. In this set-up, the state-space for a two-period model is shown in figure 1. The transition probabilities in this state-space are represented by Q.

In each time period, the stock price can undergo either of the two mutually exclusive changes. Most of the time, the price changes correspond to a state change of one unit. That is, if at time i, the state is $S_j(i)$, then at time i+1, it changes either to $S_{j+1}(i+1)$ or $S_{j-1}(i+1)$. Such changes, termed *local* price changes, correspond to the binomial changes assumed in CRR model. On rare occasions, the state changes by more than one unit. Such *non-local* changes are referred to as *jumps*. We assume that, in case of a *jump,* the stock price can jump to any of the non-adjacent states. So the structure of the state-space is that of *jumps* super-imposed on the binomial model of CRR. For simplicity, we assume that there are no dividends.

Assume that a new asset, $C_j(i)$, which is a call option on the stock, is introduced, with maturity at τ. Without loss of generality, assume that j=0. Consider the following portfolio:

$$V(i) = S_0(i)x - C_0(i) \quad (1)$$

Where $x = \frac{C_{+1}(i+1) - C_{-1}(i+1)}{S_{+1}(i+1) - S_{-1}(i+1)}$

Time 0	Time 1	Time 2
	....	....
	....	....
	$S_{+4}(1)$	$S_{+4}(2)$
	$S_{+3}(1)$	$S_{+3}(2)$
	$S_{+2}(1)$	$S_{+2}(2)$
	$S_{+1}(1)$	$S_{+1}(2)$
$S_0(0)$	$S_0(1)$	$S_0(2)$
	$S_{-1}(1)$	$S_{-1}(2)$
	$S_{-2}(1)$	$S_{-2}(2)$
	$S_{-3}(1)$	$S_{-3}(2)$
	$S_{-4}(1)$	$S_{-4}(2)$
	....	....
	....	....

(The state space for stock price dynamics over two periods)

Figure 1

The portfolio in (1) is called the delta-hedged portfolio because such a portfolio gives the same value if either of the adjacent states is realized in the next period. That is, conditional on *local* price changes in the underlying stock, the portfolio is risk-free. In what follows, for ease of reading, we suppress the subscripts and/or time index, wherever doing so is unambiguous.

If only *local* price changes happen, then, in the next period:

$$V_{+1} = V_l(i+1) = S_{+1}(i+1)x - C_{+1}(i+1) \tag{2}$$

Or

$$V_{-1} = V_l(i+1) = S_{-1}(i+1)x - C_{-1}(i+1) \tag{3}$$

Define the single period capital gain return on the underlying stock as follows:

$\Delta_k = \frac{S_{+k}(i+1)}{S(i)}$ where k=……,-2, -1 ,0, 1, 2,……..

Substituting the value of x in either (2) or (3) leads to:

$$V_{\pm}(i+1) = \frac{C_{+1}(i+1)\Delta_{-1} - C_{-1}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}} \tag{4}$$

If only *local* price changes are allowed, then the portfolio in (1) takes the value shown in (4) in the next period. That is, the delta-hedged portfolio is *locally* risk free; however, it is not *globally* risk free as *jumps* can also happen.

Now, we can specify the dynamics of awareness. Initially, assume that investors are only aware of *local* price changes. That is, they are only aware of a binomial sub-lattice in the whole state space. In syntactic representation, they are unaware of the proposition, "the delta-hedged portfolio in (1) is risky". So, they believe that the proposition "the delta-hedged portfolio is risk-free" is true. In semantic representation, if the current state is j, they are unaware of the following states (and states that can only be reached from these states):

$$S_{j+f}(i+1)\ with\ f \neq \pm 1.$$

What are the implications of this belief for the price dynamics of the call option? If the delta-hedged portfolio is believed to be risk-free then according to the principle of no-arbitrage (assets with identical state-wise payoffs should have identical state-wise returns), it should offer the same return as the risk free bond. That is,

$$V_{\pm}(i+1) = \dot{r}V(i) \tag{5}$$

Substituting (1) and (5) in (4) and simplifying leads to:

$$\dot{r}\left[\frac{C_{+1}(i+1) - C_{-1}(i+1)}{\Delta_{+1} - \Delta_{-1}}\right] - \left[\frac{C_{+1}(i+1)\Delta_{-1} - C_{-1}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}}\right] = \dot{r}C(i) \tag{6}$$

Starting from time $i = \tau$, recursive application of (6) leads to the current price of the call option.

Re-arranging (6):

$$\left(\frac{\dot{r}-\Delta_{-1}}{\Delta_{+1}-\Delta_{-1}}\right)C_{+1}(i+1)+\left(\frac{\Delta_{+1}-\dot{r}}{\Delta_{+1}-\Delta_{-1}}\right)C_{-1}(i+1)=\dot{r}C(i) \tag{7}$$

In (7), the terms in brackets in front of $C_{+1}(i+1)$ and $C_{-1}(i+1)$ are the risk neutral probabilities.

As *local* price changes and *jumps* are assumed to be mutually exclusive[3], the two are distinguished ex-post. We assume that once a jump is observed, investors become aware of the full state space. That is, they become aware that apart from *local* price changes, *jumps* can also happen. The awareness of full state space implies that investors are no longer unaware of the proposition, "the delta-hedged portfolio is risky". The delta-hedged portfolio can no longer be considered risk free by fully aware investors.

Consider the value of the delta-hedged portfolio in case of a jump. Define Y as the one period capital gain return on the stock, in case of a *jump*. That is, in case of a *jump,* the next period stock price is $S(i)Y$. The corresponding state induced by the *jump* is denoted by y. Hence, the value of the delta-hedged portfolio conditional on the *jump* is:

$$V(i+1)|\,jump=V_y(i+1)=S(i)Yx-C_y(i+1) \tag{8}$$

The delta-hedged portfolio is no longer risk free. In the case of *local* price changes, its value is risk free and is given by (4), and in the case of a *jump,* its value is risky and is given by (8). Assume that the true probability (under Q) of there being a jump is γ. Define the expectations operator with respect to the distribution of Y as E_Y. The expected value of the delta-hedged portfolio can now be written as:

$$E[V(i+1)]=\gamma E_Y\left[V_y(i+1)\right]+(1-\gamma)V_{\pm}(i+1) \tag{9}$$

As the delta-hedged portfolio can no longer be considered identical to the risk-free asset, the principle of no-arbitrage cannot be applied to determine a unique price for the call option. A call option is similar to the underlying stock, so in accordance with the principle of analogy making/generalized principle of no-arbitrage, it should offer the same expected return as the

[3] Whether *local* changes and *jumps* are mutually exclusive or not does not matter in the continuous limit.

underlying. It follows that the delta-hedged portfolio should also offer the same expected return as the underlying stock. Proposition 1 shows the recursive pricing equation that the call option must satisfy under analogy making.

Proposition 1 *If analogy making determines the price of the call option, then the following recursive pricing equation must be satisfied:*

$$\boldsymbol{(1-\gamma)\left\{C_{+1}(i+1)\left[\frac{\frac{(r+\delta-\gamma E_Y[Y])}{1-\gamma}-\Delta_{-1}}{\Delta_{+1}-\Delta_{-1}}\right]+C_{-1}(i+1)\left[\frac{\Delta_{+1}-\frac{(r+\delta-\gamma E_Y[Y])}{1-\gamma}}{\Delta_{+1}-\Delta_{-1}}\right]\right\}}$$
$$\boldsymbol{+\gamma E_Y\left[C_y(i+1)\right]=(r+\delta)C(i)} \qquad \textbf{(10)}$$

Where $\boldsymbol{\delta}$ is the risk premium on the underlying stock.

Proof.

Analogy making implies that $(r+\delta)V(i)=E[V(i+1)]$. Substituting (4) and (8) in (9) and collecting terms together leads to (10).

■

(10) differs from the corresponding recursive pricing equation in Amin (1993) due to the presence of δ, which is the risk premium on the underlying stock. If the delta-hedged portfolio in (1) is considered identical to a riskless asset, which corresponds to unawareness of a part of the state space, then according to the principle of no-arbitrage, it should offer the risk free return. In that case the pricing equation for the call option is given in (7). (7) can be obtained from (10) by making γ and δ equal to zero. If the delta-hedged portfolio in (1) is considered risky, which corresponds to full awareness of the state space, then the principle of no-arbitrage cannot be applied. However, the generalized principle of no-arbitrage, which is based on analogy making, and that says, "*Assets with similar state-wise payoffs should offer the same expected returns*", can be applied. Application of that principle results in the recursive pricing equation shown in (10).

The generalized principle of no-arbitrage or the principle of analogy making results in an arbitrage-free price for the call option. To see this, one just needs to realize that the existence of the risk neutral measure or the equivalent martingale measure is both necessary and sufficient for prices to be arbitrage-free. See Harrison and Kreps (1979). One can simply multiply payoffs with the corresponding risk neutral probabilities to get the price of an asset times the risk free rate. Proposition 2 shows the equivalent martingale measure associated with the analogy model developed here.

Proposition 2 *The equivalent martingale measure or the risk neutral pricing measure associated with the analogy model is given by:*

Risk neutral probability of a +1 change in state: $\boldsymbol{(1-\gamma_n)(q)}$

Risk neutral probability of a -1 change in state: $\boldsymbol{(1-\gamma_n)(1-q)}$

Total risk neutral probability of any other change in state or jump: $\boldsymbol{\gamma_n}$

Where $\boldsymbol{q} = \dfrac{\frac{r+\delta-\gamma E_Y[Y]}{1-\gamma}-\Delta_{-1}}{\Delta_{+1}-\Delta_{-1}}$

And $\boldsymbol{\gamma_n} = \dfrac{(1-\gamma)\dot{r}-S(i)(\dot{r}+\delta-\gamma E_Y[Y])}{(1-\gamma)E_Y[Y]-S(i)(\dot{r}+\delta-\gamma E_Y[Y])}$

Proof.

By the definition of equivalent martingale measure, the following equations must hold:

$$(1-\gamma_n)\{C_{+1}(i+1)q + C_{-1}(i+1)(1-q)\} + \gamma_n E_Y\left[C_y(i+1)\right] = \dot{r}C(i)$$

$$(1-\gamma_n)\{S_{+1}(i+1)q + S_{-1}(i+1)(1-q)\} + \gamma_n E_Y[Y]S(i) = \dot{r}S(i)$$

Substituting the values of q and γ_n in the above equations and simplifying shows that the left hand sides of the above equations are equal to $\dot{r}C(i)$ and $\dot{r}S(i)$ respectively.

■

Proposition 2 shows that the risk neutral probability of a *jump* occurring is different from the actual probability of the *jump*. Analogy making implies that the jump risk is priced causing a deviation between the actual and risk neutral probabilities.

2.1 Strongly Restricted Awareness in the Continuous Limit

In general, there are infinitely many specifications of the discrete state space described earlier that lead to the jump diffusion stochastic process in the continuous limit. As one example, see Amin (1993). For technical proofs of convergence of associated value functions, see Kushner and DeMasi (1978).

In the continuous limit, the discrete stochastic process described earlier, converges to a jump diffusion process. However, if investors are only aware of *local* price changes, then they think that the true process is geometric Brownian motion in the continuous limit. Hence, they get the type of the stochastic process wrong. We refer to such unawareness as *strongly restricted awareness* to distinguish it from a situation in which the type is know but the true parameter values are not known (*weakly restricted awareness* described in section 3).

In the continuous case presented here, we make all the assumptions made in Merton (1976) except one. Specifically, Merton (1976) assumes that the jump risk is diversifiable. Here, we assume that the jump risk is systematic and hence must be priced. To price jump risk, we do what we did for the discrete case. That is, we use the generalized principle of no-arbitrage or the principle of analogy making. However, before presenting the results, we highlight the intuition behind the convergence of the discrete model discussed earlier to the differential equation of the continuous process derived in Merton (1976). After that, we deviate from Merton (1976) and apply analogy making.

To see the intuition of this convergence, consider the value of the delta-hedged portfolio in the discrete time model considered earlier with full awareness:

Under a *local* price change:

$$V(i+1) = S_{+1}(i+1)x - C_{+1}(i+1) = S_{-1}(i+1)x - C_{-1}(i+1)$$

$=> [\Delta V]_{local} = \Delta Sx - \Delta C$

Under a *jump:*

$V(i+1) = YSx - C_y(i+1)$

$=> [\Delta V]_{jump} = (Y-1)Sx - \left(C_y(i+1) - C(i)\right)$

So, the total change in the value of the portfolio is:

$$[\Delta V]_{total} = [\Delta V]_{local} + [\Delta V]_{jump} \quad (11)$$

As in Merton (1976), assume that the size of the jump does not depend on the time interval; however, the probability of the jump depends on the time interval. This implies that as the time interval goes to zero, the probability of the jump also goes to zero; however, the size of the jump does not go to zero. For *local* changes, assume that the size of a *local* change goes to zero, as the time interval goes to zero; however, the probability of a *local* change tends to a constant. (Note, that in geometric Brownian motion the probability of movement never goes to zero as the probability tends to a constant as $dt \to 0$, however the size goes to zero as $dt \to 0$). With these assumptions, as $dt \to 0$, (11) can be written as:

$$[dV]_{total} = [dV]_{Brownian} + [dV]_{Poisson} \quad (12)$$

$$[dV]_{Brownian} = [dS]_{Brownian}x - [dC]_{Brownian} \quad (13)$$

To find out the stochastic differential equation for $[dV]_{Poisson}$, consider the process dq where in the interval dt,

$dq = 1\,; with\ probability\ \gamma dt$

$dq = 0\,; with\ probability\ (1-\gamma dt)$

It follows,

$E[dq] = \gamma dt$

And,

$Var[dq] = \gamma dt + O((dt)^2)$

Suppose a *jump* occurs in the interval dt with probability γdt. The stochastic differential equation for the *jump* process of the stock can now be written as:

$$[dS]_{jump} = (Y-1)Sdq$$

It follows (see Kushner and DeMasi (1978)),

$$[dV]_{Poisson} = (Y-1)Sxdq - \big(C(YS,t) - C(S,t)\big)dq \quad (14)$$

From Ito's lemma,

$$[dV]_{Brownian} = (\mu S dt + \sigma S dW)x - \left\{\left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 S}{\partial C^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW\right\} \quad (15)$$

Where $\mu \; and \; \sigma$ are the mean and the standard deviation of the underlying's returns respectively, and $dW = \emptyset\sqrt{dt}$. $\emptyset$ is a random draw from a normal distribution with mean zero and variance one.

Substituting (14) and (15) in (12), realizing that $x = \frac{\partial C}{\partial S}$, and suppressing the subscript 'total', leads to:

$$dV = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + (Y-1)S\frac{\partial C}{\partial S}dq - \big(C(YS,t) - C(S,t)\big)dq \quad (16)$$

As can be seen in (16), the delta-hedged portfolio is not risk-free due to the appearance of dq.

Under *partial awareness*, people are unaware of the proposition, "the delta-hedged portfolio is risky". They believe that the proposition, "the delta-hedged portfolio is risk-free", is true. Equivalently, in semantic terms, they have *restricted awareness* as they are unaware of states in which the stock price can *jump*. They incorrectly believe that the complete stochastic process is specified by the Brownian component only. That is, they incorrectly believe that the true stochastic differential equation is given by, $dV = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$, whereas the true stochastic differential equation is given in (16).

So far, our analysis of the continuous case is similar to Merton (1976). From this point onwards, we depart from Merton's assumption that the jump risk is diversifiable. Just as in the discrete case, we apply the generalized principle of no-arbitrage or analogy making to determine the price of the call option. Proposition 3 shows the partial integral differential equation (PIDE) obtained if we assume "full awareness" and the call price is determined through analogy making.

Proposition 3 ***If analogy making determines the price of a European call option, then the following partial integral differential equation (PIDE) describes the evolution of the call price:***

$$(r+\delta+\gamma)C = \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial S}[(r+\delta)S - E[Y-1]S\gamma] + E[C(YS,t)]\gamma \qquad (17)$$

Proof.

See Appendix A.

■

Note that in (17), if $\gamma = 0$ (consequently $\delta = 0$), then the Black Scholes partial differential equation is obtained. So, if people are unaware of the arrival of Poisson *jumps*, they price the option by assuming that the delta-hedged portfolio is riskless. Solving equation (17) requires some assumptions about the distribution of *jumps*. Proposition 4 shows the solution of (17) and expresses the price of a European call option as an infinite sum of a converging series.

Proposition 4 *If analogy making determines the price of a European call option, and the distribution of jumps (distribution of Y) is assumed to be log-normal with a mean of one (implying the distribution is symmetric around the current stock price) and variance of* $\boldsymbol{v^2}$*, then the price of a European call option is given by*

$$\boldsymbol{Call} = \sum_{j=0}^{\infty} \frac{e^{-\gamma(T-t)}\left(\gamma(T-t)\right)^j}{j!} \boldsymbol{Call_{AG}}\left(S, (T-t), K, r, \delta, \sigma_j\right) \quad \mathbf{(18)}$$

$$\boldsymbol{Call_{AG}}\left(S, (T-t), K, r, \delta, \sigma_j\right) = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \delta + \frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}}$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \delta - \frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}}$$

$$\sigma_j = \sqrt{\sigma^2 + v^2\left(\frac{j}{T-t}\right)}$$

$v^2 = \frac{f\sigma^2}{\gamma}$ **where f is the fraction of volatility explained by the jumps.**

Proof.

See Appendix B.

■

Corollary 4.1 *The price of a European put option is given by,*

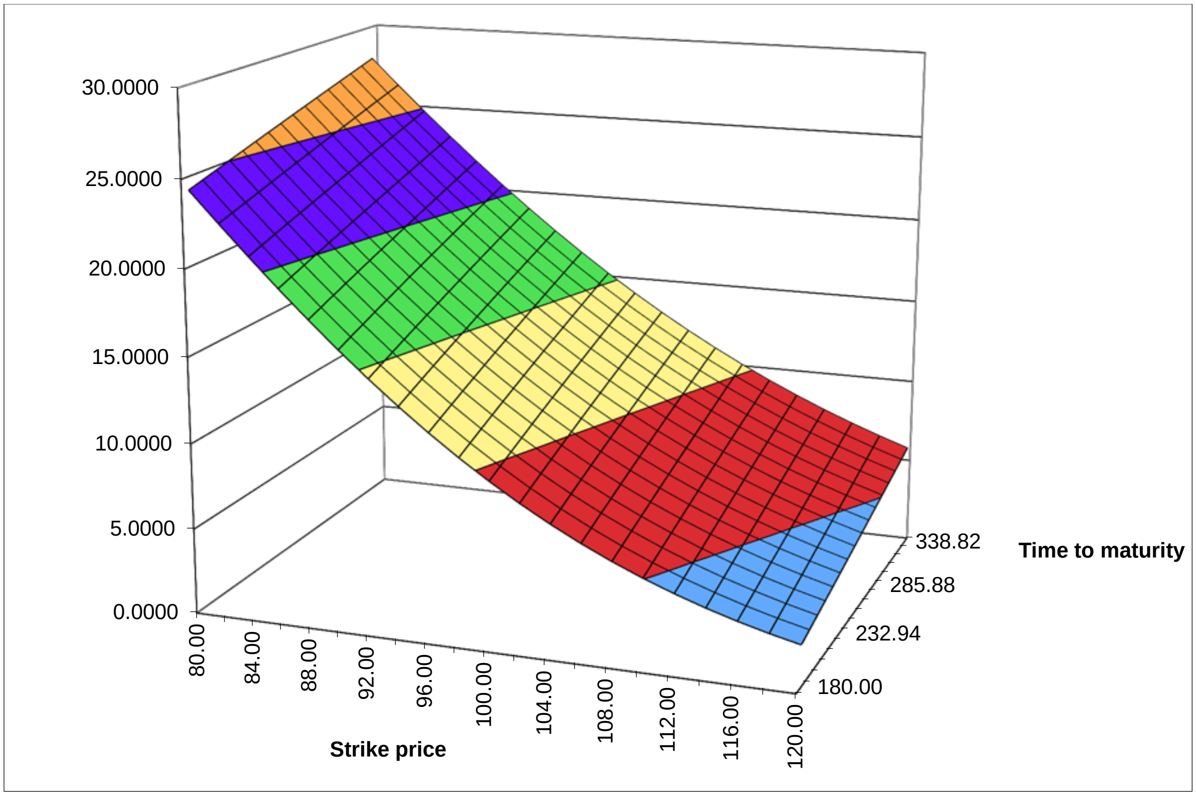
$$\boldsymbol{Put} = \sum_{j=0}^{\infty} \frac{e^{-\gamma(T-t)}\left(\gamma(T-t)\right)^j}{j!} \boldsymbol{Put_{AG}}\left(S, (T-t), K, r, \delta, \sigma_j\right) \quad \mathbf{(19)}$$

Proof.

Follows from put-call parity.

■

The formulas in (18) and (19) are identical to the corresponding Merton jump diffusion formulas except for the appearance of one additional parameter, δ, which is the risk premium on the underlying stock. $Call_{AG}$ in (18) and Put_{AG} in (19) are the analogy based formulas derived in Siddiqi (2013). So, (18) and (19) can be considered the jump diffusion generalizations of the formulas derived in Siddiqi (2013), just as Merton jump diffusion is a generalization of the Black Scholes formula. Figure 2 plots the value of a European call option against various values of the strike price, and time to maturity, as generated by (18).



Analogy based price of a Call option with jumps

$(\boldsymbol{S} = \mathbf{100}, \boldsymbol{r} = 5\%, \boldsymbol{\gamma} = \mathbf{1}\ \boldsymbol{per\ year}, \boldsymbol{\delta} = 5\%, \boldsymbol{\sigma} = \mathbf{25}\%, \boldsymbol{f} = \mathbf{10}\%)$

Figure 2

The analogy jump diffusion formula has a number of advantages over the Merton jump diffusion formula:

1) A key advantage over Merton jump diffusion formula is that the analogy formula does not assume that the jump risk is diversifiable. For index options, the risk clearly cannot be diversified away. The analogy approach provides a convenient (non utility maximization) way of pricing options in the presence of systematic jump risk, based on an empirically observed rule.

2) Merton jump diffusion formula cannot generate the implied volatility skew (monotonically declining implied volatility as a function of strike/spot) if jumps are assumed to be symmetrically distributed around the current stock price. The analogy formula can generate the skew even when the jumps are assumed to be symmetrically distributed as in (18) and (19). Assuming symmetric distribution of jumps around the current stock price, greatly simplifies the formula.

3) Even if we assume an asymmetric jump distribution around the current stock price, Merton formula, when calibrated with historical data, generates a skew which is a lot less pronounced (steep) than what is empirically observed. See Andersen and Andreasen (2002). The skew generated by the analogy formula is more pronounced (steep).

2.2 The Implied Volatility Skew

If prices are determined in accordance with the formulas given in (18) and (19) and the Black Scholes formula is used to back-out implied volatility, the skew is observed. As an example, figure 3 shows the skew generated by assuming the following parameter values:

$(\mathrm{S} = 100, \mathrm{r} = 5\%, \gamma = 1 \text{ per year}, \delta = 5\%, \sigma = 25\%, \mathrm{f} = 10\%, \mathrm{T} - \mathrm{t} = 0.5 \text{ year}).$

In figure 3, the x-axis values are various values of strike/spot, where spot is fixed at 100. Note, that the implied volatility is always higher than the actual volatility of 25%. Empirically, implied volatility is typically higher than the realized or historical volatility. As one example, Rennison and Pederson (2012) use data ranging from 1994 to 2012 from eight different option markets to calculated implied volatility from at-the-money options. They report that implied volatilities are typically higher than realized volatilities.

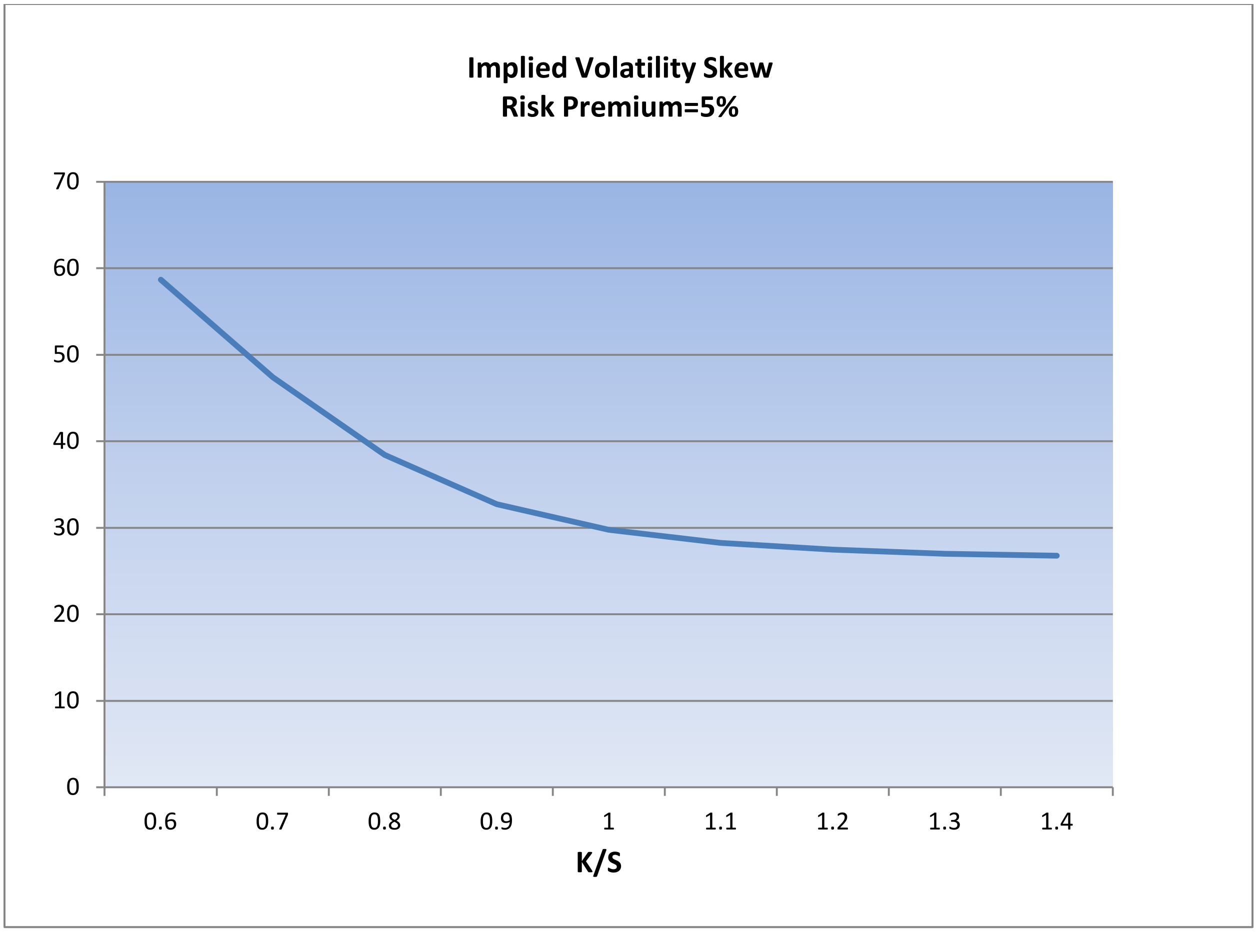


Figure 3

3. Weakly Restricted Awareness: Unawareness of True Parameter Values

If people are unaware of some states in a discrete stochastic process, then, in the continuous limit, it leads to two possibilities: 1) They may be unaware of the type of the true stochastic process. This can be termed *strongly restricted awareness*. 2) They are aware of the type of the true stochastic process but not of the true parameter values. Such awareness can be called *weakly restricted awareness*.

The first possibility has been explored in the previous section. In the previous section, we showed that *partial awareness* in which people are unaware of the proposition, "the delta-hedged portfolio is risky", is equivalent to *restricted awareness* in which people are unaware of some states. In the previous section, we assumed that the distribution of states is such that under *partial* or *restricted awareness*, the stochastic process is geometric Brownian motion, whereas the true stochastic process is jump diffusion. While under *partial* or *restricted awareness*, the principle of no-arbitrage (assets with identical state-wise payoffs must have identical state-wise returns) can be applied to price a call option, under full awareness, it cannot be applied as the 'identical asset' does not exist anymore. If the generalized principle of no-arbitrage or analogy making (assets with similar state-wise payoffs assets should offer similar state-wise returns on average) is applied, then it leads to a new option pricing formula (Analogy based jump diffusion formula), which can be considered a generalization of Merton's jump diffusion formula. If option prices are determined in accordance with the Analogy formula, and the Black Scholes formula is used to back-out implied volatility, then the implied volatility skew is observed. Hence, the sudden appearance of the skew after the crash of 1987 can be thought of as arising due to an increase in awareness in which people became aware of the proposition, "the delta-hedged portfolio is risky".

In this section, we assume that the distribution of states is such that the true stochastic process is geometric Brownian motion in the continuous limit. R*estricted awareness* in which people are unaware of at least one state or equivalently *partial awareness* in which people are unaware of the proposition, "the delta-hedged portfolio is risky" then amounts to people being unaware of the true parameter values. That is, the true type or form of the stochastic process is known, however, the true parameter values are not known.

The set-up of the model here is identical to the one described in the previous section except for the distribution of states. As before, assume a discrete lattice of states. Let $S = S_0$ at $t = 0$. Assume that at $t = \Delta t$, the following three possible state transitions can take place:

$S_0 \to S_0 + \Delta h; with\ probability\ p$

$S_0 \to S_0 - \Delta h; with\ probability\ q$

$S_0 \to S_0 - \epsilon\Delta h; with\ probability\ l$

Where $p + q + l = 1$, and $\epsilon > 0$.

Assume further:

1) S follows a Markov process i.e the probability distribution in the future depends only on where it is now.

2) At each point in time, S can only change in three ways: up by Δh or down by either Δh or $\epsilon\Delta h$.

Suppose l is very small. Assume that initially people are only aware of an up movement by Δh or a down movement by Δh. That is, they are unaware of the down movement by $\epsilon\Delta h$. It is straightforward to note that, in this set-up, people have *partial awareness* as unawareness of the third state amounts to being unaware of the proposition, "the delta-hedged portfolio is risky". That is, they believe the following proposition to be true, "the delta-hedged portfolio is risk-free".

For simplicity, and without loss of generality, we assume that the state probabilities are also misperceived such that $p'\Delta h + q'(-\Delta h) = p\Delta h + q(-\Delta h) + l(-\epsilon\Delta h)$, where the sum of the misperceived probabilities, $p' + q'$, is one. This means that the expected return on the stock is not misperceived due to restricted awareness; however, the variance is misperceived. Proposition 5 shows the connection between the true and the misperceived stochastic processes in the continuous limit.

Proposition 5 *The misperceived stochastic process under partial awareness (weak restricted awareness) given by* $\boldsymbol{dS = \mu_p S dt + \sigma_p dz}$ *corresponds to the true stochastic process given by* $\boldsymbol{dS = \mu_T S dt + \sigma_T S dz = \mu_p S dt + \sigma_p\{1 + l(\epsilon^2 - 1)\Delta h\}dz}$*.* $\boldsymbol{Where\ dz = \emptyset\sqrt{dt}}$*, and* $\emptyset \sim \boldsymbol{N(0, 1)}$*.*

Proof.

See Appendix C.

■

As awareness grows, people become aware of the true stochastic process. Consequently, they realize that the principle of no-arbitrage cannot be used to price options as the delta-hedged portfolio is no longer identical to the risk free asset. Instead, the generalized principle of no-arbitrage or analogy making is used. Proposition 6 shows the partial differential equation associated with a European call option in this set-up.

Proposition 6 *If analogy making sets the price of a European call option, the analogy option pricing partial differential Equation (PDE) is*

$$(r+\delta)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r+\delta)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2}$$

Proof.

See appendix D.

■

Proposition 7 shows the solution of the partial differential equation shown in proposition 6.

Proposition 7 *The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is*

$$C = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2) \qquad (20)$$

where $d_1 = \frac{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

Proof.

See Appendix E.

■

Corollary 7.1 *The formula for the analogy based price of a European put option is*

$$Ke^{-(r+\delta)(T-t)}N(-d_2) - SN(-d_1) \qquad (21)$$

Proof. Follows from put-call parity.

■

The formulas given in proposition 7 are identical to the corresponding Black Scholes formulas except for the appearance of δ in the new formulas, which is the risk premium on the underlying. If the Black Scholes formula represents *strongly restricted awareness*, then the correct analogy based formulas are (18) and (19). Those formulas, termed analogy based jump diffusion formulas, are a generalization of Merton's jump diffusion formulas. If the Black Scholes formula represents *weakly restricted awareness*, then the correct analogy based formulas are derived in this section and are given in proposition 7. Note, that the analogy formulas are considerably simpler (given in (20) and (21)) if we assume *weakly restricted awareness*.

3.1 Implied Volatility Skew

If option prices are determined in accordance with the formulas given in (20) and (21), and the Black Scholes model is used to back-out implied volatility, then the skew is observed as figure 4 shows.

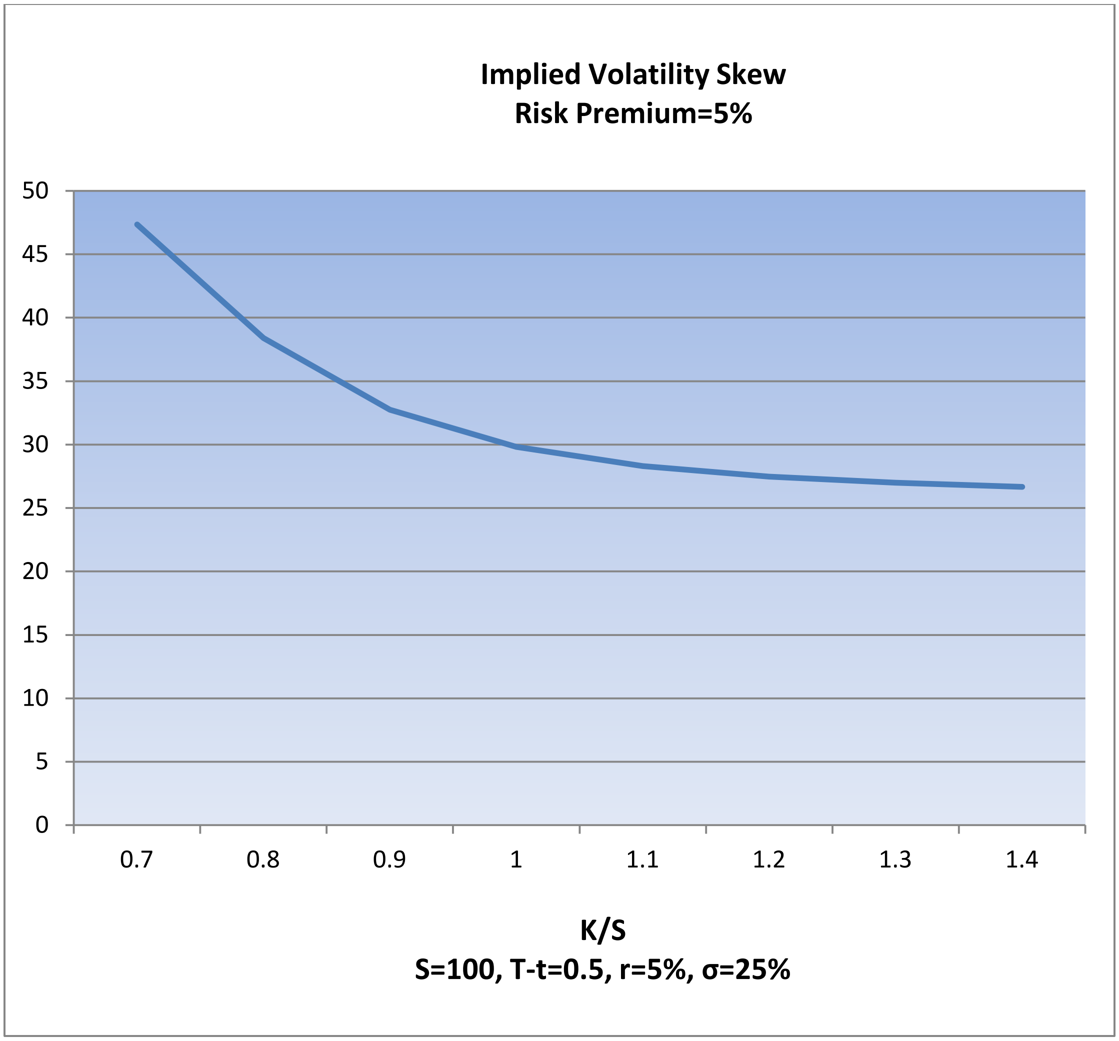


Figure 4

4. Strongly Restricted or Weakly Restricted Awareness?

Does the Black Scholes option pricing model represent *strongly restricted awareness* or *weakly restricted awareness*? If it represents strongly restricted awareness, then the analogy based formulas (under full awareness) are given in (18) and (19). If it represents *weakly restricted awareness*, then the relevant analogy based formulas (under full awareness) are given in (20) and (21).

The major disadvantage of (18) and (19) is that they are more complex than (20) and (21) as they have two additional parameters. However, they also have a key advantage over (20) and (21) as they can capture both the implied volatility skew as well as the implied volatility smile. In reality, for index option, the skew is observed (implied volatility always declines as K/S rises). However, for options on individual stocks both the skew as well as the smile (implied volatility of deep in-the-money as well as deep-out-of-the money options is higher than the implied volatility of at-the-money options) is observed. Figure 5 shows one instance of an implied volatility smile generated by (18). Here, we assume that the risk premium on the underlying stock is 1%, and the fraction of volatility explained by jumps is 40%. The rest of the parameters are the same as in figure 5.

In general, the skew generated by (18) and (19) turns into a smile as the risk premium on the underlying falls (approaches the risk-free rate). This is consistent with empirical evidence as individual stocks are typically considered to have lower risk premiums compared to the risk premiums on indices. It seems that for individual stock options, where both the skew and the smile is observed, and for currency and commodity options, (18) and (19) are required. However, for index options, simpler formulas given in (20) and (21) are appropriate.

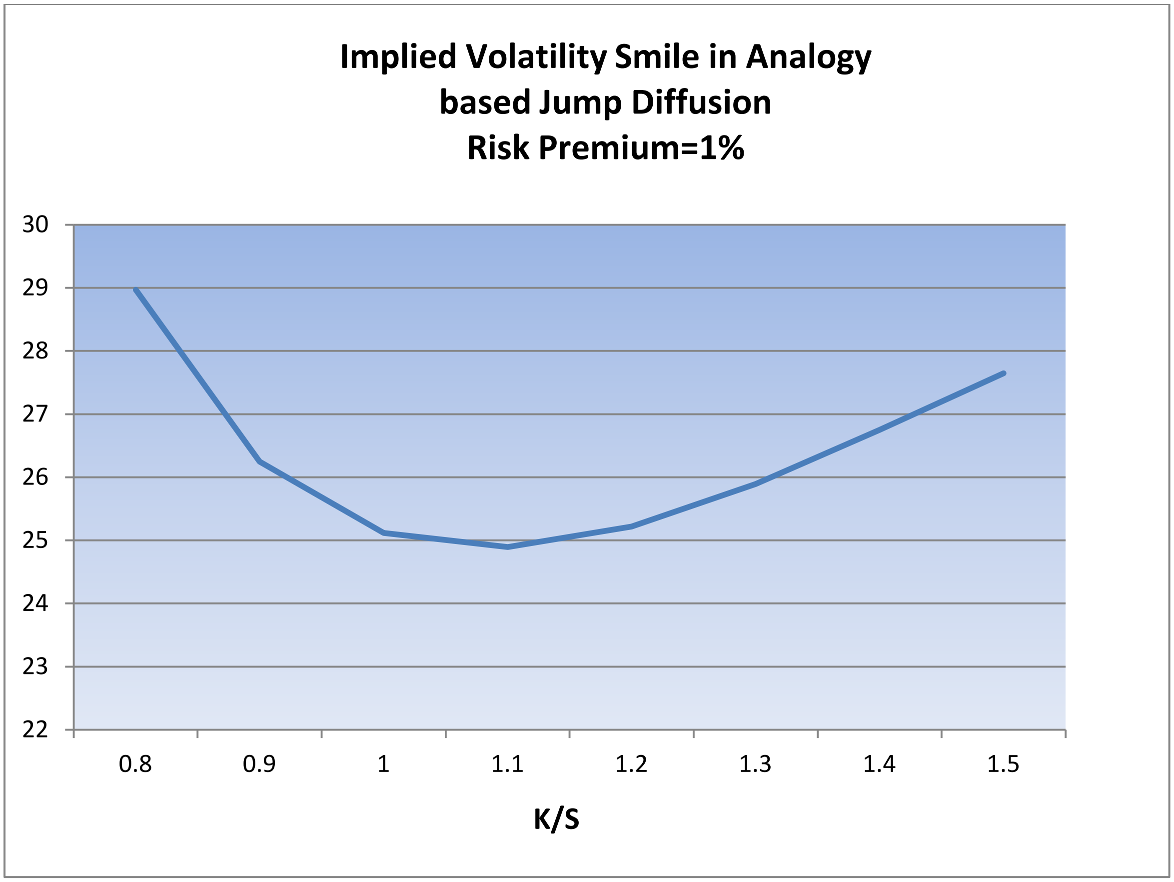


Figure 5

5. Conclusions

It is interesting to try and model forgetfulness in this set-up. Forgetfulness can be described as an opposite process to growing awareness. Suppose one is initially aware that 'a delta-hedged portfolio is risky' but observes for a considerable length of time that 'a delta-hedged portfolio is risk-free'. He may be tempted to think that the stochastic process has changed and the only possible states are those in which 'a delta-hedged portfolio is risk-free'. Thinking of awareness in the context of a stochastic process allows sufficient flexibility to model forgetfulness.

It is also interesting to compare the value of a signal under full-awareness vs. the value of the same signal under *partial awareness*. Quiggin (2013b) shows that the sum of *value of awareness* and *value of information,* appropriately defined, is a constant. As awareness corrects either an undervaluation or an overvaluation, positive and negative information signals have different impacts post-awareness when compared with pre-awareness impacts. Perhaps, this asymmetry in impacts can be exploited to devise an econometric test that would help in identifying events around which awareness changed.

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Appendix D

In the analogy case with full awareness, the expected growth rate of the portfolio $S\frac{\partial C}{\partial S} - C$ is $r + \delta$. To deduce the analogy based PDE consider:

$$V = S\frac{\partial C}{\partial S} - C$$

$$\Rightarrow dV = dS\frac{\partial C}{\partial S} - dC$$

Where $dS = uSdt + \sigma SdW$ and by Ito's Lemma $dC = \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW$

$$E[dV] = E[dS]\frac{\partial C}{\partial S} - E[dC]$$

$$=> (r+\delta)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r+\delta)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$=> (r+\delta)C = (r+\delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}$$

Appendix E

The analogy based PDE derived in Appendix D can be solved by converting to heat equation and exploiting its solution.

Start by making the following transformation:

$$\tau = \frac{\sigma^2}{2}(T - t)$$

$$x = ln\frac{S}{K} => S = Ke^{x}$$

$$C(S,t) = K \cdot\ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} - K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (A1) and writing $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$, we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1)\frac{\partial c}{\partial x} - \tilde{r}c \qquad (B1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S-K, 0\}\ becomes\ c(x,0) = max\{e^{x} - 1, 0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - \tilde{r} - \beta)u + \left(2\alpha + (\tilde{r} - 1)\right)\frac{\partial u}{\partial x} \qquad (B2)$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4}$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (B3)$$

With the initial condition:

$$u(x_0, 0) = max\{(e^{(1-\alpha)x_0} - e^{-\alpha x_0}), 0\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0) dx_0$$

Change variables: $= \frac{x_0 - x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0 \; iff \; x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\left(x + z\sqrt{2\tau}\right) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau\frac{(\tilde{r}+1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi} - \sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r} + 1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r} - 1)$ and $f = \frac{\tilde{r}-1}{2} x + \tau \frac{(\tilde{r}-1)^2}{4}$

The analogy based European call pricing formula is obtained by recovering original variables:

$$Call = SN(d_1) - Ke^{-(r+\delta)(T-t)} N(d_2)$$

Where $\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K) + (r + \delta + \frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ $and\ \boldsymbol{d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right) + \left(r + \delta - \frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$

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