

2014

FINANCE

**RISK AND SUSTAINABLE
MANAGEMENT GROUP
WORKING PAPER SERIES**

TITLE:

Analogy Making and the Puzzles of
Index Option Returns and Implied
Volatility Skew: Theory and
Empirical Evidence

AUTHOR:

Hammad Siddiqi

Working Paper: F14_4

**Schools of
Economics and
Political Science
The University of
Queensland
St Lucia
Brisbane
Australia 4072**

**Web:
www.uq.edu.au**

Analogy Making and the Puzzles of Index Option Returns and Implied Volatility Skew: Theory and Empirical Evidence

Hammad Siddiqi¹

Working Paper

This Version: July 8, 2014

The University of Queensland

h.siddiqi@uq.edu.au

Constantinides et al (2013) put forward a number of empirical findings regarding leverage adjusted S&P 500 index option returns. Their findings are puzzling in the context of the Black-Scholes-Merton Option Pricing Model and the Capital Asset Pricing Model. Experimental evidence as well as the opinions of experienced market professionals indicate that call options are valued in analogy with the underlying stock. In this article, the implications of such analogy making for option pricing are explored, and the resulting analogy based option pricing model is put forward. In a one period binomial setting, I show the conditions under which arbitrage profits cannot be made against the analogy makers ensuring their survival. I show that the analogy model is consistent with the empirical findings in Constantinides et al (2013). Furthermore, the analogy model generates the implied volatility skew. Two predictions of the analogy model are also empirically tested and are found to be strongly supported in the data.

JEL Classification: G13; G12

Keywords: Option Pricing; Analogy Making; Leverage Adjusted Returns; Risk Premium; Implied Volatility Skew; Implied Volatility

¹ I am grateful to John Quiggin, Simon Grant, Begona Dominguez, and the seminar participants in Queensland Macro workshop for helpful comments and suggestions. All errors and omissions are due to the author.

Analogy Making and the Puzzles of Index Option Returns and Implied Volatility Skew: Theory and Empirical Evidence

Constantinides, Jackwerth and Savov (2013) uncover a number of puzzling facts regarding leverage adjusted index option returns. They find that over a period ranging from April 1986 to January 2012, the average percentage monthly returns of leverage-adjusted index call and put options are decreasing in the ratio of strike to spot. They also find that leverage adjusted put returns are larger than the corresponding leverage adjusted call returns. Furthermore, they find that leverage adjusted put returns (especially for out-of-the-money puts) tend to fall as time to expiry increases, whereas the leverage adjusted call returns (especially for out-of-the-money calls) tend to rise as time to expiry increases. These findings are in contradiction with the Black-Scholes-Merton Option Pricing Model (and the Capital Asset Pricing Model) which requires that the leverage adjusted returns should not change with the ratio of strike to spot as well as option maturity.

There is strong experimental evidence that participants mentally co-categorize a call option with its underlying and price it in analogy with the underlying. See Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011). In Rockenbach (2004), the price of the underlying stock as well as the possible payoffs along with their respective probabilities was known to participants. That is, participants knew the expected return available on the underlying objectively. Rockenbach (2004) finds that the hypothesis that “participants price a call option by equating its expected return with the expected return available on the underlying stock”, explains the experimental data best. From this point onwards, we refer to this hypothesis as the analogy model. Rockenbach (2004) finds that the results are robust to changing the state probabilities in a binomial setting. Siddiqi (2012) and Siddiqi (2011) extend this result and test this finding with multiple states and for various levels of option moneyness.

In the controlled environment of laboratory experiments, it is possible to objectively fix the expected return on the underlying and make it common knowledge. However, in the real world, investors are likely to have different subjective assessments of the expected return on the underlying stock. A typical call buyer is likely to be more optimistic about the future prospects of the underlying stock than the marginal investor in the underlying stock. This is because both pessimistic and optimistic views about the stock influence the belief of the marginal investor, whereas call buying is a strategy of an optimistic investor. Also, more optimistic investors are likely to self-select into

higher strike calls. So, when investors value call options in analogy with the underlying, their differing expectations should be reflected in call prices. Hence, with analogy making, not only one expects average call return to be larger than average return on the underlying stock, one also expects it to increase with the strike price.

As the above discussion indicates, the analogy model makes the same qualitative prediction regarding simple expected returns as other option pricing models such as the Black Scholes model. To strongly differentiate them, one needs to find and test other predictions of the analogy model. In this article, I put forward two such predictions: 1) When call is in-the-money, the difference between leverage adjusted put and call returns should decrease as the ratio of strike to spot increases at all levels of option maturity 2) The difference between leverage adjusted put and call returns should decrease as maturity increases at all levels of the ratio of strike to spot.

By using the dataset developed in Constantinides et al (2013) that spans nearly 26 years, I test both predictions and find strong empirical support. I also show that the analogy model is consistent with the empirical findings in Constantinides et al (2013) regarding leverage adjusted option returns. Furthermore, the analogy model generates the implied volatility skew. Hence, the analogy model provides a unified explanation for a number of option pricing puzzles.

Is their existing field evidence of mental co-categorization of a call option with its underlying? Market professionals with decades of experience typically consider a call option to be a surrogate for the underlying and advise their clients to consider replacing the underlying with a call option if they expect at least the same return from it as from the underlying stock.² It seems natural that such mental framing influences option valuation. This is even more plausible given the fact that an option is typically valued relative to the price of the underlying. Hence, the return on the underlying is a natural reference point for forming expectations about an option.

If such mental framing or analogy making influences option valuation, shouldn't rational arbitrageurs make money at the expense of analogy makers by taking appropriate positions in a call option and in the corresponding replicating portfolio in accordance with the Black-Scholes approach? No, because the Black-Scholes model ignores transaction costs, and when transaction costs are allowed, no matter how small they are, the perfect replication argument fails in continuous

² As illustrative examples, see the following:

<http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772>,
<http://www.investingblog.org/archives/194/deep-in-the-money-options/>,
<http://www.triplescreenmethod.com/TradersCorner/TC052705.asp>,
<http://daytrading.about.com/od/stocks/a/OptionsInvest.htm>

time as transaction costs grow without bound. It is well known that there is no non-trivial portfolio that replicates a call option in the presence of transaction costs. See Soner, Shreve, and Cvitanic (1995). In discrete time, the transaction costs are bounded, however, a no-arbitrage interval is created. If the analogy price lies within the interval, analogy makers cannot be arbitrated away. I show the conditions under which this happens in a one period binomial setting.

There is a large and growing body of literature in economics and finance that acknowledges the importance of mental framing for investment decisions. Two examples from the asset pricing stream are Shefrin and Statman (2000) and Barberis and Huang (2001) among others. This paper is methodologically closest to Shefrin (2008) who provides a systematic treatment of how behavioural assumptions impact the pricing kernel at the heart of modern asset pricing theory.

This paper is organized as follows. Section 2 discusses the puzzling empirical findings in Constantinides et al (2013) in the context of the Capital Asset Pricing Model and the Black Scholes Model. Section 3 illustrates the analogy approach with numerical examples in a binomial setting. Section 4 considers the general binomial case with analogy making. The analogy formula in continuous time is derived in section 5. Section 6 shows that the analogy model provides a unified explanation for the findings in Constantinides et al (2013). It also shows that the analogy model generates the implied volatility skew. Section 7 puts forward the predictions of the analogy model, which are tested in Section 8. Section 9 concludes.

2. Puzzling Empirical Findings in Constantinides, Jackwerth, and Savov (2013)

It is useful to start with a review of the predictions of the capital asset pricing model (CAPM) and the Black Scholes model (BSM) regarding the leverage adjusted index option returns. Intuitively, leverage adjustment dilutes the beta risk of an option by combining it with a risk free asset. Beta risk of options differs widely across strike and maturity. Leverage adjustment combines each option with a risk-free asset in such a manner that the overall beta risk becomes equal to the beta risk of the underlying stock. Specifically, the weight of the option in the portfolio is equal to its inverse price elasticity w.r.t the underlying stock's price:

$$\beta_{portfolio} = \Omega^{-1} \times \beta_{call} + (1 - \Omega^{-1}) \times \beta_{riskfree} \quad (1)$$

where $\Omega = \frac{\partial Call}{\partial Stock} \times \frac{Stock}{Call}$ (i.e price elasticity of call w.r.t the underlying stock)

$$\begin{aligned}
\beta_{call} &= \Omega \times \beta_{stock} \\
\beta_{riskfree} &= 0 \\
\Rightarrow \beta_{portfolio} &= \beta_{stock}
\end{aligned} \tag{2}$$

As (2) shows, CAPM predicts that all leverage adjusted option portfolios (with the same underlying stock) have the same beta regardless of differing strikes and maturities, and that beta is equal to the beta of the underlying stock. Hence, if one believes in CAPM, the same return is expected from them, which is equal to the return on the underlying stock.

The Black Scholes model makes the same prediction regarding leverage adjusted option returns:

$$\Omega^{-1} \cdot \frac{1}{dt} \left[\frac{E[dC]}{C} \right] + (1 - \Omega^{-1})r$$

where C is the price of a call option, and r is the risk-free rate.

$$\Rightarrow \Omega^{-1} \cdot \left\{ \frac{1}{dt} \left[\frac{E[dC]}{C} - r \right] \right\} + r \tag{3}$$

$$E[dC] = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt \tag{4}$$

where μ = *instantaneous expected return*, σ = *instantaneous st. dev. of returns* of the underlying stock, and S is the price of the underlying stock.

The Black Scholes PDE is

$$rC = rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \tag{5}$$

Substituting (5) and (4) in (3):

$$\begin{aligned}
&\Omega^{-1} \{ (\mu - r) \Omega \} + r \\
\Rightarrow & \text{Leverage Adjusted Return} = \mu
\end{aligned}$$

The above illustration assumes a call option; however, the illustration with a put option is very similar and leads to the same conclusion (except that one takes a short position in a put option in

constructing leverage adjusted put portfolios) Hence, irrespective of option strike and maturity, the expected return from a leverage adjusted portfolio should be equal to the expected return from the underlying stock if the Black Scholes model is correct.

Table 1 (which is table 3 from Constantinides et al (2013)) shows monthly average returns on leverage adjusted S&P 500 index options. Options data ranges from April 1986 to January 2012. Table 1 shows that instead of being equal to 0.86% (average monthly return on S&P 500 index) irrespective of strike and maturity, the returns exhibit a number of interesting features:

- 1) They are decreasing in the ratio of strike to spot.
- 2) Put returns are larger than corresponding call returns
- 3) For call options, they typically increase as expiry increases (especially when strike of call is bigger than the underlying index's price).
- 4) For put options, they typically decrease as expiry increases (especially when strike of put is smaller than the underlying index's price).

Table 1 (Table 3 from Constantinides et al (2013))

Average percentage monthly returns of the leverage adjusted portfolios. April 1986 to January 2012. For comparison, average monthly return on S&P 500 index is 0.86%.												
	Call						Put					
K/S	90	95%	100%	105%	110%	Hi-Lo	90	95%	100%	105%	110%	Hi-Lo
Average monthly returns												
30 days	0.49	0.42	0.21	0.03	-0.02	-0.51	2.18	1.66	1.07	0.80	0.75	-1.43
(s.e)	0.24	0.24	0.24	0.23	0.22	0.17	0.36	0.32	0.29	0.27	0.26	0.20
90 days	0.51	0.44	0.37	0.31	0.21	-0.30	1.15	1.10	0.91	0.81	0.74	-0.40
(s.e)	0.24	0.24	0.24	0.24	0.24	0.11	0.33	0.31	0.29	0.27	0.27	0.14
90-30	0.03	0.02	0.16	0.28	0.23		-1.04	-0.55	-0.16	0.00	-0.01	
(s.e)	0.02	0.02	0.03	0.06	0.11		0.11	0.07	0.03	0.02	0.02	

The above patterns are quite intriguing. Note, that the BSM-CAPM prediction is that they should be equal to 0.86%, which is the return on S&P 500 index.

3. Analogy Making: A Numerical Illustration

Consider an investor in a two state-two asset complete market world with one time period marked by two points in time: 0 and 1. The two assets are a stock (S) and a risk free bond (B). The stock has a price of \$140 today (time 0). Tomorrow (time 1), the stock price could either go up to \$200 (the red state) or go down to \$94 (the blue state). Each state has a 50% chance of occurring. There is a riskless bond that has a price of \$100 today. Its price stays at \$100 at time 1 implying a risk free rate of zero. Suppose a new asset “A” is introduced to him. The asset “A” pays \$100 in cash in the red state and nothing in the blue state. How much should the investor be willing to pay for this new asset?

Finance theory provides an answer by appealing to the principle of no-arbitrage: *assets with identical state-wise payoffs must have the same price* or equivalently *assets with identical state-wise payoffs must have the same state-wise returns*. Consider a portfolio consisting of a long position in 0.943396 of S and a short position in 0.886792 of B. In the red state, 0.943396 of S pays \$188.6792 and one has to pay \$88.6792 due to shorting of 0.886792 of B earlier resulting in a net payoff of \$100. In the blue state, 0.943396 of S pays \$88.6792 and one has to pay \$88.6792 on account of shorting 0.886792 of B previously resulting in a net payoff of 0. That is, payoffs from 0.943396S-0.886792B are identical to payoffs from “A”. As the cost of 0.943396S-0.886792B is \$43.39623, it follows that the no-arbitrage price for “A” is \$43.39623.

When simple tasks such as the one described above are presented to participants in a series of experiments, instead of the no-arbitrage argument, they seem to rely on analogy-making to figure out their willingness to pay. See Rockenbach (2004), Siddiqi (2011), and Siddiqi (2012). Instead of trying to construct a replicating portfolio which is identical to asset “A”, people find an actual asset similar to “A” and price “A” in analogy with that asset. They rely on the principle of analogy: *assets with similar state-wise payoffs should offer the same state-wise returns on average*, or equivalently, *assets with similar state-wise payoffs should have the same expected return*.

Asset “A” is similar to asset S. It pays more when asset S pays more and it pays less when asset S pays less. Expected return from S is $1.05 \left(\frac{0.5 \times 200 + 0.5 \times 94}{140} \right)$. According to the principle of analogy, A’s price should be such that it offers the same expected return as S. That is, A is valued at \$47.61905. Note that asset “A” is equivalent to a call option on “S” with a strike price of 100.

In the above example, there is a gap of \$4.22281 between the no-arbitrage price and the analogy price. Rational investors should short “A” and buy “0.943396S-0.886792B”. However, transaction costs are ignored in the example so far.

Let’s see what happens when a symmetric proportional transaction cost of only 1% of the price is applied when assets are traded. That is, both a buyer and a seller pay a transaction cost of 1% of the price of the asset traded. Unsurprisingly, the composition of the replicating portfolio changes. To successfully replicate a long call option that pays \$100 in cash in the red state and 0 in the blue state with transaction cost of 1%, one needs to buy 0.952925 of S and short 0.878012 of B. In the red state, 0.952925S yields \$188.6792 net of transaction cost ($200 \times 0.952925 \times (1 - 0.01)$), and one has to pay \$88.6792 to cover the short position in B created earlier ($0.878012 \times 100 \times (1 + 0.01)$). Hence, the net cash generated by liquidating the replicating portfolio at time 1 is \$100 in the red state. In the blue state, the net cash from liquidating the replicating portfolio is 0. Hence, with a symmetric and proportional transaction cost of 1%, the replicating portfolio is “0.952925S-0.878012B”. The cost of setting up this replicating portfolio inclusive of transaction costs at time 0 is \$47.82044, which is larger than the price the analogy makers are willing to pay: \$47.61905. Hence, arbitrage profits cannot be made at the expense of analogy makers by writing a call and buying the replicating portfolio. The given scheme cannot generate arbitrage profits unless the call price is greater than \$47.82044

Suppose one is interested in doing the opposite. That is, buy a call and short the replicating portfolio to fund the purchase. Continuing with the same example, the relevant replicating portfolio (that generates an outflow of \$100 in the red state and 0 in the blue state) is “-0.934056S+0.89575B”. The replicating portfolio generates \$41.1928 at time 0, which leaves \$38.98937 after time 0 transaction costs in setting up the portfolio are paid. Hence, in order for the scheme to make money, one needs to buy a call option at a price less than \$38.98937.

Effectively, transaction costs create a no-arbitrage interval (38.98937, 47.82044). As the analogy price lies within this interval, arbitrage profits cannot be made at the expense of analogy makers in the example considered.

Some cognitive scientists argue that analogy making forms the core of cognition and it is the fuel and fire of thinking (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, “[...] at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known.”

(Hofstadter and Sander (2013), Prologue page1).

The recognition of analogy making as an important decision principle is not new. Hume wrote in 1748, “From causes which appear similar, we expect similar effects. This is the sum of all our experimental conclusions”. (Hume 1748, Section IV). Similar ideas have been expressed in economic literature by Keynes (1921), Selten (1978), and Cross (1983) among others. To our knowledge, two formal approaches have been proposed to incorporate analogy making into economics: 1) case based decision theory of Gilboa and Schmeidler (2001) in which preferences are determined by the cases in a decision maker’s memory and their similarity with the decision problem being considered, and 2) coarse thinking/analogy making model of Mullainathan, Schwartzstein, and Shleifer (2008) in which expectations about an attribute are formed by co-categorizing a situation with analogous situations and transferring the information content of the attribute across co-categorized situations. The approach in this paper, if broadly interpreted, relates to the model of Mullainathan et al (2008). The attribute of concern here is return on a call option, which is influenced by the return on the underlying as investors co-co-categorize a call with the underlying stock.

3.1 Analogy Making: A Two Period Binomial Example with Delta Hedging

Consider a two period binomial model. The parameters are: Up factor=2, Down factor=0.5, Current stock price=100, Risk free interest rate per binomial period=0, Strike price=30, and the probability of up movement=0.5. It follows that the expected gross return from the stock per binomial period is 1.25 ($0.5 \times 2 + 0.5 \times 0.5$).

The call option can be priced both via analogy as well as via no-arbitrage argument. The no-arbitrage price is denoted by C_R whereas the analogy price is denoted by C_A . Define $x_R = \frac{\Delta C_R}{\Delta S}$ and $x_A = \frac{\Delta C_A}{\Delta S}$ where the differences are taken between the possible next period values that can be reached from a given node.

Figure 1 shows the binomial tree and the corresponding no-arbitrage and analogy prices. Two things should be noted. Firstly, in the binomial case considered, before expiry, the analogy price is always larger than the no-arbitrage price. Secondly, the delta hedging portfolios in the two cases $Sx_R - C_R$ and $Sx_A - C_A$ grow at different rates. The portfolio $Sx_A - C_A$ grows at the rate equal to the expected return on stock per binomial period (which is 1.25 in this case). In the analogy

case, the value of delta-hedging portfolio when the stock price is 100 is $17.06667 (100 \times 0.98667 - 81.6)$. In the next period, if the stock price goes up to 200, the value becomes $21.33333 (200 \times 0.98667 - 176)$. If the stock price goes down to 50, the value also ends up being equal to $21.33333 (50 \times 0.98667 - 28)$. That is, either way, the rate of growth is the same and is equal to 1.25 as $17.06667 \times 1.25 = 21.33333$. Similarly, if the delta hedging portfolio is constructed at any other node, the next period return remains equal to the expected return from stock. It is easy to verify that the portfolio $Sx_R - C_R$ grows at a different rate which is equal to the risk free rate per binomial period (which is 0 in this case).

Exp. Ret 1.25
 Up Prob. 0.5
 Up 2
 Down 0.5
 Risk-Free r 0
 Strike 30

Stock Price	400
$Call_R$	370
$Call_A$	370

Stock Price	200
x_R	1
B	-30
$Call_R$	170
x_A	1
$Call_A$	176

Stock Price	100
x_R	0.977778
B	-25.5556
$Call_R$	72.22222
x_A	0.986667
$Call_A$	81.6

Stock Price	100
$Call_R$	70
$Call_A$	70

Stock Price	50
x_R	0.933333
B	-23.3333
$Call_R$	23.33333
x_A	0.933333
$Call_A$	28

Stock Price	25
$Call_R$	0
$Call_A$	0

Figure 1

4. Analogy Making: The Binomial Case

Consider a two state world. The equally likely states are Red, and Blue. There is a stock with prices X_1 , and X_2 corresponding to states Red, and Blue respectively, where $X_1 > X_2$. The state realization takes place at time T . The current time is time t . We denote the risk free discount rate by r . That is, there is a riskless bond that has a price of B in both states with a price of $\frac{B}{1+r}$ today. For simplicity and without loss of generality, we assume that $r = 0$ and $T - t = 1$. The current price of the stock is S such that $X_1 > S > X_2$. We further assume that $S < \frac{X_1+X_2}{2}$. That is, the stock price reflects a positive risk premium. In other words, $S = f \cdot \frac{X_1+X_2}{2}$ where $f = \frac{1}{1+r+\delta}$.³ δ is the risk premium reflected in the price of the stock. As we have assumed $r = 0$, it follows that $f = \frac{1}{1+\delta}$.

Suppose a new asset which is a European call option on the stock is introduced. By definition, the payoffs from the call option in the two states are:

$$C_1 = \max\{(X_1 - K), 0\}, C_2 = \max\{(X_2 - K), 0\} \quad (6)$$

Where K is the striking price, and C_1 , and C_2 , are the payoffs from the call option corresponding to Red, and Blue states respectively.

How much is an analogy maker willing to pay for this call option?

There are two cases in which the call option has a non-trivial price: 1) $X_1 > X_2 > K$ and 2) $X_1 > K > X_2$

The analogy maker infers the price of the call option, P_c , by equating the expected return from the call to the return he expects from holding the underlying stock:

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \quad (7)$$

³ In general, a stock price can be expressed as a product of a discount factor and the expected payoff if it follows a binomial process in discrete time (as assumed here), or if it follows a geometric Brownian motion in continuous time.

For case 1 ($X_1 > X_2 > K$), one can write:

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$\Rightarrow P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \quad (8)$$

Substituting $S = f \cdot \frac{X_1 + X_2}{2}$ in (8):

$$P_c = S - Kf \quad (9)$$

The above equation is the one period analogy option pricing formula for the binomial case when call expires in-the-money in both states.

The corresponding no-arbitrage price P_r is (from the principle of no-arbitrage):

$$P_r = S - K \quad (10)$$

For case 2 ($X_1 > K > X_2$), the analogy price is:

$$P_c = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \quad (11)$$

And, the corresponding no-arbitrage price is:

$$P_r = \frac{X_1 - K}{X_1 - X_2} (S - X_2) \quad (12)$$

Proposition 1 *The analogy price is larger than the corresponding no-arbitrage price if a positive risk premium is reflected in the price of the underlying stock and there are no transaction costs.*

Proof.

For case 1, when $X_1 > X_2 > K$, the results follow from a direct comparison of (9) and (10).

For case 2, when $X_1 > K > X_2$, the spectrum of possibilities is further divided into three sub-classes and the results are proved for each sub-class one by one. The three sub-classes are: (i) $K = \frac{X_1+X_2}{2}$, (ii) $X_2 < K < \frac{X_1+X_2}{2}$, and (iii) $X_1 > K > \frac{X_1+X_2}{2}$.

Case 2 sub-class (i): $K = \frac{X_1+X_2}{2}$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2} (S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = \frac{X_1+X_2}{2}$ above and simplify, it follows that $f \geq 1$, which is a contradiction as $f < 1$ if the risk premium is positive.

Case 2 sub-class (ii): $X_2 < K < \frac{X_1+X_2}{2}$ or equivalently $K = g \frac{X_1+X_2}{2}$ where $\frac{2X_2}{X_1+X_2} < g < 1$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2} (S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = g \frac{X_1+X_2}{2}$ above and simplify, it follows that $X_1 \leq X_2$, which is a contradiction.

Case 2 sub-class (iii): $X_1 > K > \frac{X_1+X_2}{2}$ or equivalently $K = g \frac{X_1+X_2}{2}$ where $1 < g < \frac{2X_1}{X_1+X_2}$

Similar logic as used in the case above leads to a contradiction: $X_1 \leq X_2$.

Hence, the analogy price must be larger than the no-arbitrage price if the risk premium is positive and there are no transaction costs.

■

Suppose there are transaction costs, denoted by “c”, which are assumed to be symmetric and proportional. That is, if the stock price is S, a buyer pays $S(1 + c)$ and a seller receives $S(1 - c)$. Similar rule applies when the bond or the option is traded. That is, if the bond price is B, a buyer pays $B(1 + c)$ and a seller receives $B(1 - c)$. We further assume that the call option is cash settled. That is, there is no physical delivery.

Introduction of the transaction cost does not change the analogy price as the expected returns (unrealized) on call and on the underlying stock are proportionally reduced. However, the cost of replicating a call option changes. The total cost of successfully replicating a long position in the call option by buying the appropriate replicating portfolio and then liquidating it in the next period to get cash (as call is cash settled) is:

$$\left(\frac{X_1 - K}{X_1 - X_2}\right) \left\{ \frac{S}{1-c} - \frac{X_2}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{X_2}{1+c} \right\} \quad \text{if } X_1 > K > X_2 \quad (13)$$

$$\left\{ \frac{S}{1-c} - \frac{K}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{K}{1+c} \right\} \quad \text{if } X_1 > X_2 > K \quad (14)$$

The corresponding inflow from shorting the appropriate replicating portfolio to fund the purchase of a call option is:

$$\left(\frac{X_1 - K}{X_1 - X_2}\right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} \quad \text{if } X_1 > K > X_2 \quad (15)$$

$$\left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\} \quad \text{if } X_1 > X_2 > K \quad (16)$$

Proposition 2 shows that if transaction costs exist and the risk premium on the underlying stock is within a certain range, the analogy price lies within the no-arbitrage interval. Hence, riskless profit cannot be earned at the expense of analogy makers.

Proposition 2 *In the presence of transaction costs, analogy makers cannot be arbitrated out of the market if the risk premium on the underlying stock satisfies:*

$$0 \leq \delta \leq \frac{(1-c)(1+c)}{(1-c)^2 - 2\frac{S}{K}c(1+c)} - 1 \quad \text{if } X_1 > X_2 > K \quad (17)$$

$$0 \leq \delta \leq$$

$$\frac{K(X_1^2 - X_2^2)(1 - c^2)}{2} - \frac{1}{X_2(X_1 - K)(X_1 + X_2)(1 - c)^2 - S\{(1 + c)^2(X_1^2 - X_2^2) - X_1(X_1 - X_2)(1 - c^2)\}}$$

if $X_1 > K > X_2$ (18)

Proof.

If $X_1 > X_2 > K$ then there is no-arbitrage if the following holds:

$$\left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\} \leq S - Kf \leq \left\{ \frac{S}{1-c} - \frac{K}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{K}{1+c} \right\}$$

Realizing that $S - Kf \geq S - K > \left\{ \frac{S}{1+c} - \frac{K}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{K}{1-c} \right\}$ if $\delta \geq 0$ and simplifying

$$S - Kf \leq \left\{ \frac{S}{1-c} - \frac{K}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{K}{1+c} \right\} \text{ leads to inequality (17).}$$

If $X_1 > K > X_2$ then there is no-arbitrage if the following holds:

$$\begin{aligned} \left(\frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} &\leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \\ &\leq \left(\frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1-c} - \frac{X_2}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{X_2}{1+c} \right\} \end{aligned}$$

Realizing that

$$\begin{aligned} \left(\frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1+c} - \frac{X_2}{1-c} \right\} - c \left\{ \frac{S}{1+c} + \frac{X_2}{1-c} \right\} &\leq \\ \frac{X_1 - K}{X_1 - X_2} (S - X_2) &\leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \text{ if } \delta \geq 0 \end{aligned}$$

And simplifying $S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \leq \left(\frac{X_1 - K}{X_1 - X_2} \right) \left\{ \frac{S}{1-c} - \frac{X_2}{1+c} \right\} + c \left\{ \frac{S}{1-c} + \frac{X_2}{1+c} \right\}$ leads to (18).

■

Intuitively, when transaction costs are introduced, there is no unique no-arbitrage price. Instead, a whole interval of no-arbitrage prices comes into existence. Proposition 2 shows that for reasonable parameter values, the analogy price lies within this no-arbitrage interval in a one period binomial model. As more binomial periods are added, the transaction costs increase further due to the need for additional re-balancing of the replicating portfolio. In the continuous limit, the total transaction cost is unbounded. Reasonably, arbitrageurs cannot make money at the expense of analogy makers in the presence of transaction costs ensuring that the analogy makers survive in the market.

It is interesting to consider the rate at which the delta-hedged portfolio grows under analogy making. Proposition 3 shows that under analogy making, the delta-hedged portfolio grows at a rate $\frac{1}{f} - 1 = r + \delta$. This is in contrast with the Black Scholes Merton/Binomial Model in which the growth rate is equal to the risk free rate, r .

Proposition 3 *If analogy making determines the price of the call option, then the corresponding delta-hedged portfolio grows with time at the rate of $\frac{1}{f} - 1$.*

Proof.

Case 1: $X_1 > X_2 > K$

Delta-hedged portfolio is $Sx - C$. In this case, $x = 1$, $S = f \cdot \frac{X_1 + X_2}{2}$, and $C = S - Kf$

If the red state is realized, $S - C$ changes from Kf to K . If the blue state is realized $S - C$ also changes from Kf to K . Hence, the growth rate is equal to $\frac{1}{f} - 1$ in either state.

Case 2: $X_1 > K > X_2$

Delta-hedged portfolio is $Sx - C$. In this case, $x = \frac{X_1 - K}{X_1 - X_2}$, $S = f \cdot \frac{X_1 + X_2}{2}$, and

$$C = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f$$

Consider three sub-classes and prove the result for each: (i) $K = \frac{X_1 + X_2}{2}$, (ii) $X_2 < K < \frac{X_1 + X_2}{2}$, and (iii) $X_1 > K > \frac{X_1 + X_2}{2}$. For the first sub-class the delta-hedged portfolio changes from the initial value of $f \frac{X_2}{2}$ to $\frac{X_2}{2}$ in both the red and the blue states. Hence, the growth rate is equal to $\frac{1}{f} - 1$ in either state. For the second and third sub-classes, the delta-hedged portfolio changes from

$\frac{f\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)}$ to $\frac{\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)}$ in both red and blue states. Hence, the growth rate is equal to $\frac{1}{f} - 1$.

■

Corollary 3.1 *If there are multiple binomial periods then the growth rate of the delta-hedged portfolio per binomial period is $\frac{1}{f} - 1$.*

In continuous time, the difference in the growth rates of the delta-hedged portfolio under analogy making and under the Black Scholes/Binomial model leads to an option pricing formula under analogy making which is different from the Black Scholes formula. The continuous time formula is presented in the next section.

5. Analogy Making: The Continuous Case

We maintain all the assumptions of the Black-Scholes-Merton model except one. We allow for transaction costs whereas the transaction costs are ignored in the Black-Scholes-Merton model. As is well known, introduction of the transaction costs invalidates the replication argument underlying the Black Scholes formula. See Soner, Shreve, and Cvitanic (1995). As seen in the last section, transaction costs have no bearing on the analogy argument as they simply reduce the expected return on the call and on the underlying stock proportionally.

Proposition 4 shows the analogy based partial differential equation under the assumption that the underlying follows geometric Brownian motion, which is the limiting case of the discrete binomial model. We also explicitly allow for the possibility that different marginal investors determine prices of calls with different strikes. This is reasonable as call buying is a bullish strategy with more optimistic buyers self-selecting into higher strikes.

Proposition 4 *If analogy makers set the price of a European call option, the analogy option pricing partial differential Equation (PDE) is*

$$(r + \delta_K)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r + \delta_K)S + \frac{\partial^2 C}{\partial S^2} \frac{\sigma^2 S^2}{2}$$

Where δ_K is the risk premium that a marginal investor in the call option with strike 'K' expects from the underlying stock.

Proof.

See Appendix A ■

Just like the Black Scholes PDE, the analogy option pricing PDE can be solved by transforming it into the heat equation. Proposition 5 shows the resulting call option pricing formula for European options without dividends under analogy making.

Proposition 5 *The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is $C = SN(d_1) - Ke^{-(r+\delta_K)}N(d_2)$ where $d_1 =$*

$$\frac{\ln(S/K) + (r + \delta_K + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\ln(\frac{S}{K}) + (r + \delta_K - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Proof.

See Appendix B.

■

Corollary 5.1 *The formula for the analogy based price of a European put option is*

$$Ke^{-r(T-t)}\{1 - e^{-\delta_K(T-t)}N(d_2)\} - SN(-d_1)$$

Proof. Follows from put-call parity.

■

As proposition 5 shows, the analogy formula is exactly identical to the Black Scholes formula except for the appearance of δ_K , which is the risk premium that a marginal investor in the call option with strike K expects from the underlying stock. Note, that full allowance is made for the possibility that such expectations vary with strike price as more optimistic investors are more likely to self-select into higher strike calls.

6. Analogy Making: A Unified Explanation for Leverage Adjusted Index Option Returns

If analogy making determines call prices, then the behavior of leverage adjusted call and put returns should be a lot different than their predicted behavior under the Black-Scholes assumptions. For call options:

$$\Omega^{-1} \cdot \frac{1}{dt} \left[\frac{E[dC]}{C} \right] + (1 - \Omega^{-1})r \quad (19)$$

$$\text{where } E[dC] = \left((r + \delta_K)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt \quad (20)$$

According to the analogy based PDE:

$$(r + \delta_K)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} (r + \delta_K)S + \frac{\partial^2 C}{\partial S^2} \frac{\sigma^2 S^2}{2} \quad (21)$$

Substituting (20) and (21) in (19) and simplifying leads to:

$$\text{Leverage Adjusted Call Return} = \Omega^{-1}(\delta_K) + r \quad (22)$$

(22) describes the behaviour of leverage adjusted call option returns under analogy making. It is obvious from the expression that the behavior of leverage adjusted returns w.r.t strike and maturity is crucially dependent on the behavior of Ω w.r.t strike and maturity.

By using S&P 100 and OEX option data ranging from 1986 to 1995, Coval and Shumway (2001) find that the level of average option returns is exceedingly small given the level of systematic risk at all levels of strike prices considered. In particular, a glance at Table 1 and table 2 in Coval and Shumway (2001) also confirms that the change in average index option returns is exceedingly small given the substantial change in systematic risk caused by increasing the striking price.

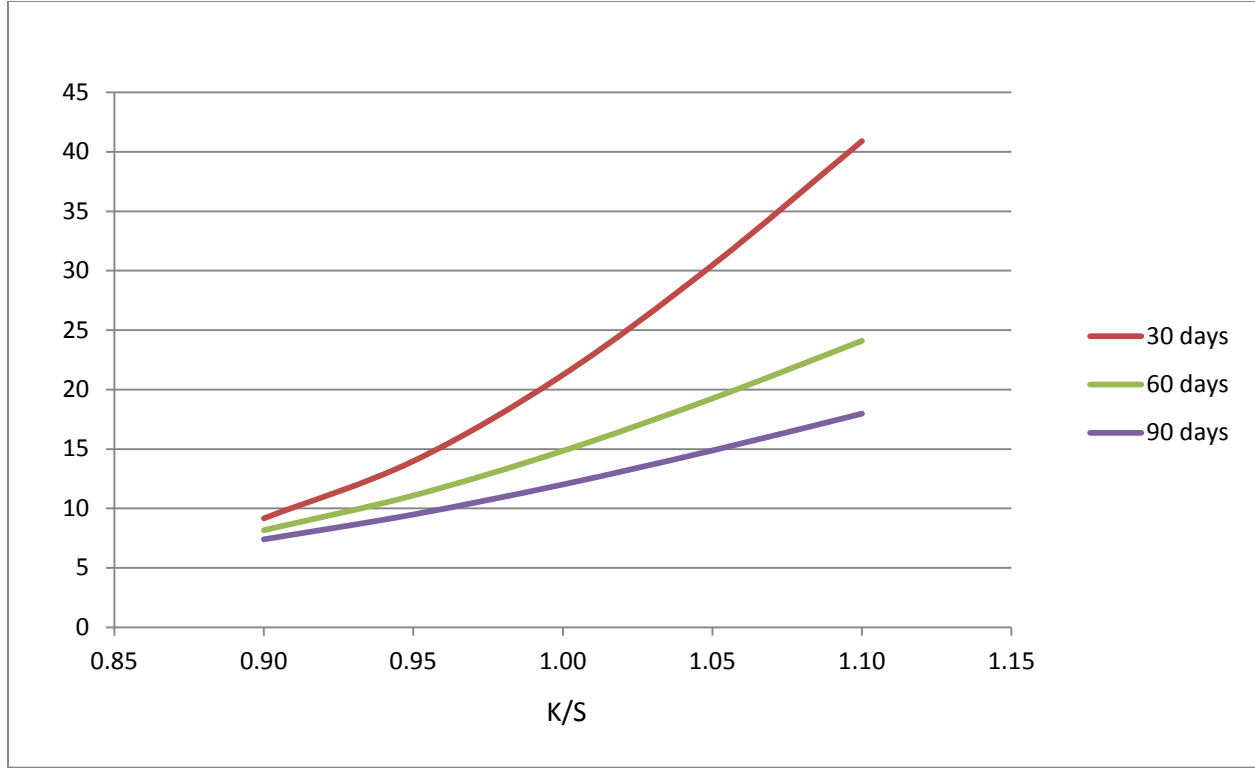
Given that Ω changes substantially with striking price and takes very large values for out-of-the money calls, one can safely conclude that the behavior of leverage adjusted call option returns as strike price changes, is determined by the behavior of call's price elasticity w.r.t the underlying stock's price (Ω). That is, the change in Ω dwarfs the change in δ_K .

Under the analogy model:

$$\Omega = \frac{SN(d_1)}{c} \text{ where } d_1 = \frac{\ln(S/K) + (r + \delta_K + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (23)$$

Figure 2 is a representative plot of Ω as it varies with the ratio of strike to spot and maturity. As can be seen, Ω increases very rapidly as the ratio of strike to spot increases. Ω falls as maturity increases, and quite rapidly so for out-of-the-money calls.

Call Price Elasticity (Ω)



Values used to generate the graph: $S = 100, \sigma = 20\%, \mu = 8\%$

Figure 2

Hence, according to the analogy model, the leverage adjusted call option returns (given in (22)) should fall as the ratio of strike to spot increases. As Ω falls with expiry and quite rapidly so for out-of-the-money calls, the leverage adjusted call option returns (given in (22)) should increase with expiry and quite prominently so for out-of-the-money calls. Hence, the analogy model is qualitatively consistent with the findings in Constantinides et al (2013).

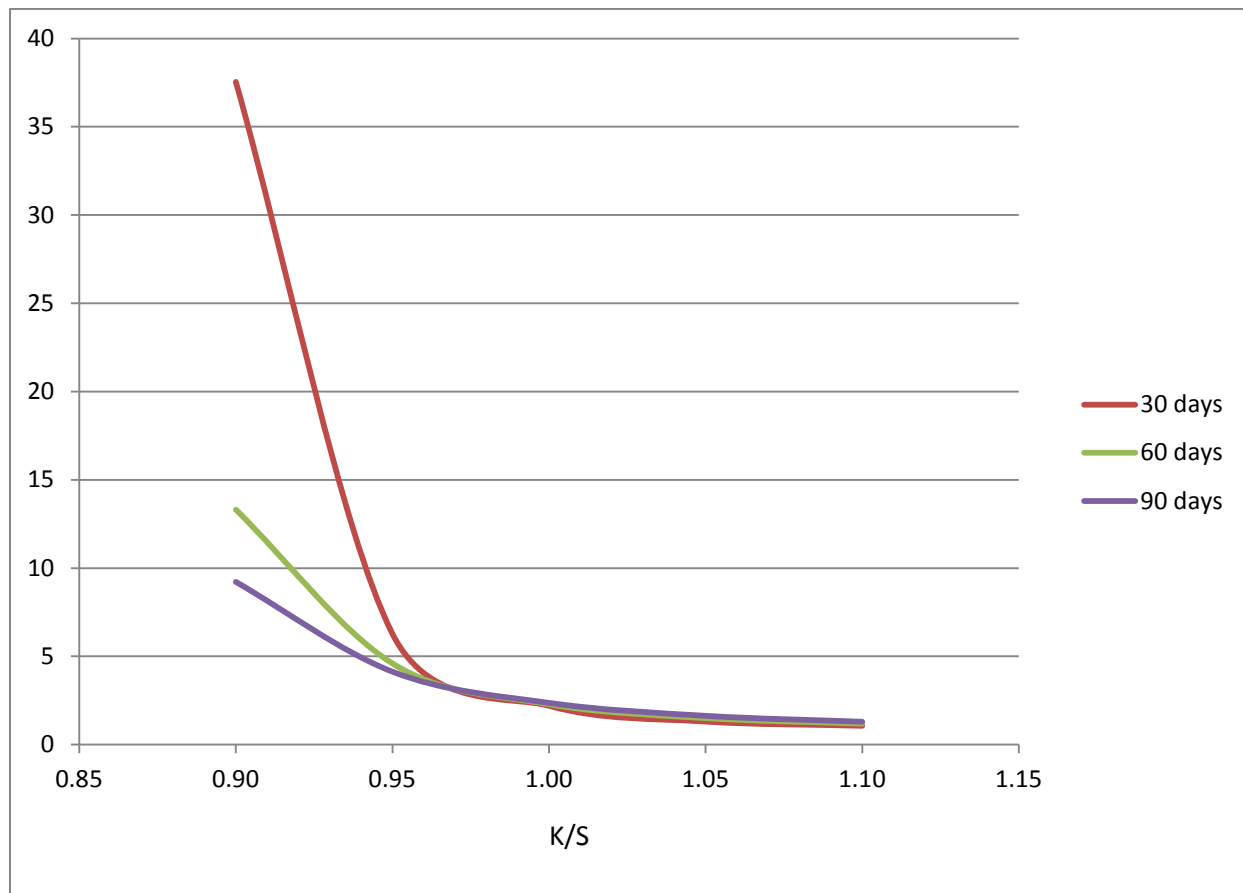
By using put-call parity, one can derive the expression for leverage adjusted put option returns:

$$\text{Leverage Adjusted Put Return} = r + \delta_K \left\{ \frac{S-C}{S(1-N(d_1))} \right\} \quad (24)$$

Clearly, (24) is greater than (22), hence, under analogy making, the leverage adjusted put returns are larger than the corresponding leverage adjusted call returns, in accordance with the empirical findings in Constantinides et al (2013).

The behavior of leverage adjusted put returns is driven by the behavior of $\frac{S-C}{S(1-N(d_1))}$. Figure 3 shows the representative behavior of $\frac{S-C}{S(1-N(d_1))}$ as the ratio of strike to spot and time to maturity changes.

The Behavior of $\frac{S-C}{S(1-N(d_1))}$



Values used to generate the graph: $S = 100, \sigma = 20\%, \mu = 8\%$

Figure 3

As can be seen from figure 3, according to the analogy model, the leverage adjusted put option returns (given in (24)) should fall as the ratio of strike to spot increases. Figure 3 also shows that for out-of-the-money puts, $\frac{S-C}{S(1-N(d_1))}$ falls with expiry very rapidly. So, according to the analogy model, the leverage adjusted put option returns should fall with expiry quite prominently for out-of-the-

money puts. Hence, the analogy model is qualitatively consistent with the findings in Constantinides et al (2013) regarding leverage adjusted put option returns.

Clearly, the analogy model can generate all of the four key features identified in section 2, which are observed in leverage adjusted index option returns.

6.1 Analogy Making and the Implied Volatility Skew

All the variables in the Black Scholes formula are directly observable except for the standard deviation of the underlying stock's returns. So, by plugging in the values of observables, the value of standard deviation can be inferred from market prices. This is called implied volatility. If the Black Scholes formula is correct, then the implied volatility values from options that are equivalent except for the strike prices should be equal. However, in practice, for equity index options, a skew is observed in which in-the-money call options' (out-of-the money puts) implied volatilities are higher than the implied volatilities from at-the-money and out-of-the-money call options (in-the-money puts).

The analogy approach developed here provides an explanation for the skew. If the analogy formula is correct, and the Black Scholes model is used to infer implied volatility then skew arises as table 2 shows. As table 2 shows, implied volatility skew is seen if the analogy formula is correct, and the Black Scholes formula is used to infer implied volatility. Notice that in the example considered, difference between the Black Scholes price and the analogy price is quite small even when implied volatility gets more than double the value of actual volatility. Note that in table 2, we have used the same risk premium on the underlying stock for different strikes for simplicity. One can easily modify the illustration to have increasing risk premia on the underlying stock with increasing striking prices and still get the skew.

Table 2				
Implied Volatility Skew				
Underlying Stock's Price=100, Volatility=20%, Risk Premium on the Underlying=5%, Time to Expiry=0.06 year				
K	Black Scholes Price	Analogy Price	Difference	Implied Volatility
105	0.5072	0.5672	0.06	20.87
100	2.160753	2.326171	0.165417	21.6570
95	5.644475	5.901344	0.25687	24.2740
90	10.30903	10.58699	0.277961	31.8250
85	15.26798	15.53439	0.266419	42.9400
80	20.25166	20.50253	0.250866	54.5700

Figure 4 is the graphical illustration of table 2.

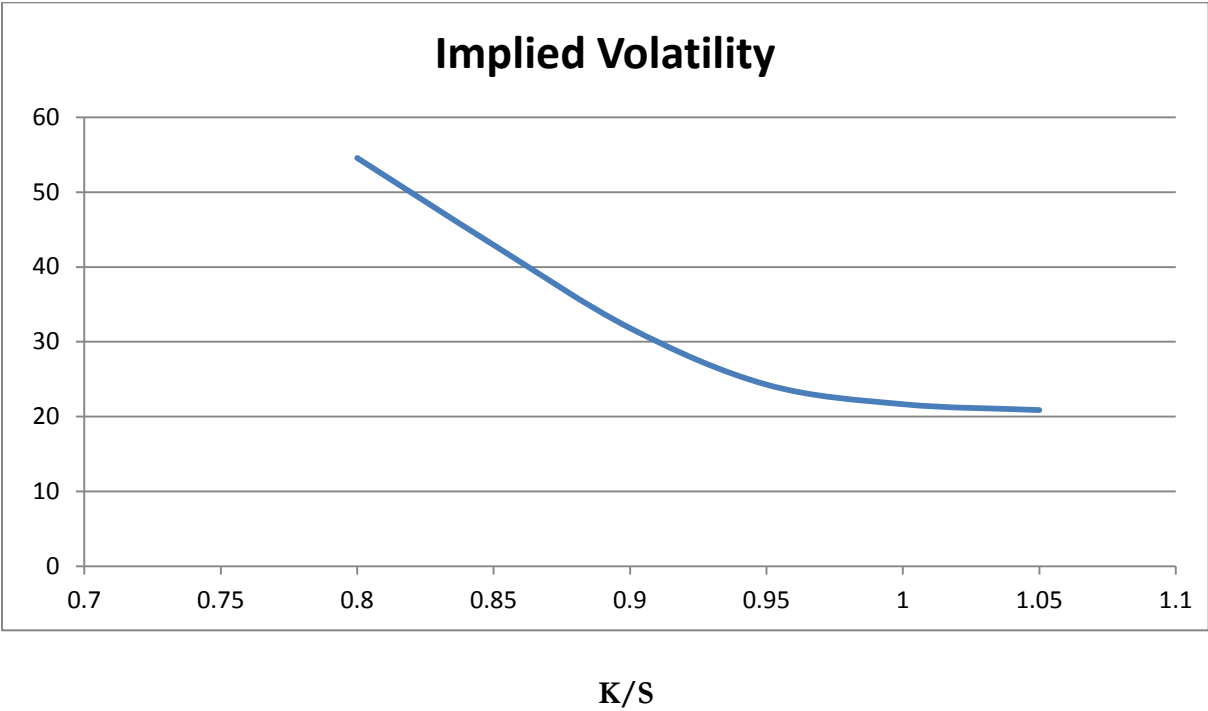


Figure 4

7. Predictions of the Analogy Model

By subtracting (22) from (24), one obtains:

$$\text{Put minus Call Return (Leverage Adjusted)} = \delta_K \left\{ \frac{s-c}{s(1-N(d_1))} - \frac{c}{SN(d_1)} \right\} \quad (25)$$

When call is in-the-money, $\frac{s-c}{s(1-N(d_1))}$ falls very rapidly with the ratio of strike to spot, as figure 3 shows. For out-of-the-money call range, the decline is a lot slower. As the ratio of strike to spot increases, $\Omega = \frac{SN(d_1)}{c}$ rises, as figure 2 shows, meaning that $\Omega^{-1} = \frac{c}{SN(d_1)}$ falls. Moreover, δ_K is expected to increase as the ratio of strike to spot rises. The last two effects have an increasing influence on the difference given in (25), whereas the first effect has a decreasing influence. Looking at figures 2 and 3, it is clear that for in-the-money call range, the change in $\frac{s-c}{s(1-N(d_1))}$ is a lot bigger than the change in Ω^{-1} . For out-of-the-money call range, the change in $\frac{s-c}{s(1-N(d_1))}$ is comparable to the change in Ω^{-1} . The following prediction of the analogy model directly follows:

Prediction 1 *When call is in-the-money, the difference between leverage adjusted put and call returns should fall as the ratio of strike to spot increases, at all levels of expiry.*

$\frac{s-c}{s(1-N(d_1))}$ falls as time to expiry increases, as figure 3 shows. $\Omega = \frac{SN(d_1)}{c}$ falls with time to expiry as well, as figure 2 shows. In other words, $\Omega^{-1} = \frac{c}{SN(d_1)}$ rises with time to expiry. Moreover, δ_K (which is the risk premium on the underlying that a marginal investor in a call option with strike K expects), is likely to fall as time to expiry increases as more optimistic investors tend to buy shorter term options.

Considering all of the above, it follows that the value generated by (25) should fall as time to expiry increases. The following prediction of the analogy model follows:

Prediction 2: *If the analogy model is correct, then the difference between the leverage adjusted put and call returns should fall as time to expiry increases at all levels of the ratio of strike to spot.*

In the next section, both predictions are tested by using nearly 26 years of options data put together in Constantinides et al (2013).

8. Empirical Evidence

The dataset used in this paper is available at http://www.wiwi.uni-konstanz.de/fileadmin/wiwi/jackwerth/Working_Paper/Version325_Return_Data.txt

The construction of this dataset is described in detail in Constantinides et al (2013). It is almost 26 years of monthly data on leverage adjusted S&P-500 index option returns ranging from April 1986 to January 2012.

For prediction 1, Wilcoxon signed rank test is used as it allows a direct observation by observation comparison of two time series. The following procedure is adopted:

- 1) The dataset has the following ratios of strikes to spot: 0.9, 0.95, 1.0, 1.05, and 1.10. For each value of strike to spot, the difference between leverage adjusted put and call returns is calculated.
- 2) Pair-wise comparisons are made between time series of 0.9 and 0.95, 0.95 and 1.0, 1.0 and 1.05, and 1.05 and 1.10. Such comparisons are made for each level of maturity: 30 days, 60 days, or 90 days.
- 3) The first time series in each pair is dubbed series1, and the second time series in each pair is dubbed series 2. That is, for the pair, 0.9 and 0.95, 0.9 is Series 1, and 0.95 is Series 2.
- 4) For each pair, if prediction 1 is true, then Series 1 > Series 2. This forms the alternative hypothesis in the Wilcoxon signed rank test, which is tested against the null hypothesis: Series 1 = Series 2

Table 2			
Put minus Call Return (Monthly)	Put minus Call Return (Monthly)	Maturity (days)	Wilcoxon Signed Rank Test
Leverage Adjusted	Leverage Adjusted		Null Hypothesis: Series 1=Series 2
(April 1986 to January 2012)	(April 1986 to January 2012)		Alternate Hypothesis: Series 1>Series 2
Series 1 Strike (%spot)	Series 2 Strike (%spot)		P-Value
0.9	0.95	30	5.62883E-14
0.95	1	30	2.33147E-14
1	1.05	30	0.095264801
1.05	1.1	30	0.378791967
0.9	0.95	60	2.23715E-06
0.95	1	60	2.08904E-11
1	1.05	60	1.31059E-09
1.05	1.1	60	0.978440796
0.9	0.95	90	0.002029759
0.95	1	90	2.84604E-08
1	1.05	90	0.10253709
1.05	1.1	90	0.696743837

Table 2 shows the results for prediction 1. As can be seen from the table, when call is in-the-money, the difference between leverage adjusted put and call returns falls with strike to spot at all levels of expiry (Series 1 is greater than Series 2). Hence, null hypothesis is rejected, in accordance with prediction 1. As expected, the p-values are quite large for out-of-the-money call range, so null cannot be rejected for out-of-the-money call range.

To test prediction 2, the procedure adopted is very similar to the one used for prediction 1:

- 1) For each level of strike to spot, the following pairwise comparisons are made: 30 days vs 60 days, 60 days vs 90 days, 30 days vs 90 days.
- 2) The first time series in each pair is dubbed Series 1, and the second time series is labelled Series 2. If prediction 2 is true, then Series 1 > Series 2. This forms the alternate hypothesis against the null: Series 1 = Series 2.
- 3) Wilcoxon signed rank test is conducted for each pair.

Table 3			
Put minus Call Return (Monthly)	Put minus Call Return (Monthly)	Strike (%spot)	Wilcoxon Signed Rank Test
Leverage Adjusted	Leverage Adjusted		Null Hypothesis: Series 1 = Series 2
(April 1986 to January 2012)	(April 1986 to January 2012)		Alternate Hypothesis: Series 1 > Series 2
Series 1 Maturity (days)	Series 2 Maturity (days)		P-Value
30	60	0.9	0.0000
60	90	0.9	0.0000
30	90	0.9	0.0000
30	60	1.1	0.0125
60	90	1.1	0.0033
30	90	1.1	0.0020

Table 3 shows the results for prediction 2. As can be seen from the table, in accordance with prediction 2, as time to expiry increases, the difference between leverage adjusted put and call returns falls at all levels of the ratio of strike to spot.

9. Conclusions

The notion that analogy making is important for human decision making is not new in economics. Case based decision theory of Gilboa and Schmeidler (2001) in which preferences are determined by the cases in a decision maker’s memory and their similarity with the decision problem being considered is one example of explicit recognition of the importance of analogy making. Another example is the coarse thinking model of Mullainathan, Schwartzstein, and Shleifer (2008) in which expectations about an attribute are formed by co-categorizing a situation with analogous situations and transferring the information content of the attribute across co-categorized situations. Some cognitive scientists and psychologists have been arguing that analogy making is the fuel and fire of thinking (see Hofstadter and Sander (2013)). Similar thoughts have been expressed in economics literature by Keynes (1921), Selten (1978), and Cross (1983) among others.

In the context of this literature, the experimental finding that participants value a call option in analogy with the underlying stock is not surprising at all. Given the human propensity for analogy making, and the fact that a call option derives its value from the underlying stock, any other finding would be a lot more surprising. As mentioned in the introduction, if one looks at the investment advice generated by market professionals, one finds that many market professionals with decades of experience are valuing a call option in analogy with the underlying stock

In this article, the implications of analogy making for option prices are explored and a behavioral option pricing model that follows is put forward. The behavioral model is consistent with the empirical findings in Constantinides et al (2013) regarding leverage adjusted index option returns. The analogy model also generates the implied volatility skew. Two predictions of the analogy model are also tested and found to be well supported.

References

- Amin, K., 1993. Jump diffusion option valuation in discrete time. *Journal of Finance* 48, 1833-1863.
- Babcock, L., & Loewenstein, G. (1997). "Explaining bargaining impasse: The role of self-serving biases". *Journal of Economic Perspectives*, 11(1), 109–126.
- Babcock, L., Wang, X., & Loewenstein, G. (1996). "Choosing the wrong pond: Social comparisons in negotiations that reflect a self-serving bias". *The Quarterly Journal of Economics*, 111(1), 1–19.
- Bakshi G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. *Journal of Finance* 52, 2003-2049.
- Ball, C., Torous, W., 1985. On jumps in common stock prices and their impact on call option pricing. *Journal of Finance* 40, 155-173.
- Belini, F., Fritelli, M. (2002), "On the existence of minimax martingale measures", *Mathematical Finance* 12, 1–21.
- Black, F., Scholes, M. (1973). "The pricing of options and corporate liabilities". *Journal of Political Economy* 81(3): pp. 637-65
- Black, F., (1976), "Studies of stock price volatility changes. Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economic Statistics Section, 177–181.
- Bollen, N., and R. Whaley. 2004. Does Net Buying Pressure Affect the Shape of Implied Volatility Functions? *Journal of Finance* 59(2): 711–53
- Bossaerts, P., Plott, C. (2004), "Basic Principles of Asset Pricing Theory: Evidence from Large Scale Experimental Financial Markets". *Review of Finance*, 8, pp. 135-169.
- Carpenter, G., Rashi G., & Nakamoto, K. (1994), "Meaningful Brands from Meaningless Differentiation: The Dependence on Irrelevant Attributes," *Journal of Marketing Research* 31, pp. 339-350
- Christensen B. J., and Prabhala, N. R. (1998), "The Relation between Realized and Implied Volatility", *Journal of Financial Economics* Vol.50, pp. 125-150.
- Christie A. A. (1982), "The stochastic behavior of common stock variances: value, leverage and interest rate effects", *Journal of Financial Economics* 10, 4 (1982), pp. 407–432
- Cross, J. G. (1983), *A Theory of Adaptive Economic Behavior*. New York: Cambridge University Press.
- Davis, M.H.A. (1997), "Option pricing in incomplete markets, *Mathematics of Derivative Securities*", Cambridge University Press, edited by M.A.H. Dempster and S.R. Pliska, 216–226.
- DeBondt, W. and Thaler, R. (1985). "Does the Stock-Market Overreact?" *Journal of Finance* 40: 793-805
- Derman, E. (2003), "The Problem of the Volatility Smile", talk at the Euronext Options Conference available at http://www.ederman.com/new/docs/euronext-volatility_smile.pdf
- Derman, E. (2002), "The perception of time, risk and return during periods of speculation". *Quantitative Finance Vol. 2*, pp. 282-296.

- Derman, E., Kani, I., & Zou, J. (1996), "The local volatility surface: unlocking the information in index option prices", *Financial Analysts Journal*, 52, 4, 25-36
- Duffie, D., C. Huang. 1985. Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica* 53 1337–1356.
- Duan, Jin-Chuan, and Wei Jason (2009), "Systematic Risk and the Price Structure of Individual Equity Options", *The Review of Financial studies*, Vol. 22, No.5, pp. 1981-2006.
- Duan, J.-C., 1995. The GARCH option pricing model. *Mathematical Finance* 5, 13-32.
- Dumas, B., Fleming, J., Whaley, R., 1998. Implied volatility functions: empirical tests. *Journal of Finance* 53, 2059-2106.
- Edelman, G. (1992), *Bright Air, Brilliant Fire: On the Matter of the Mind*, New York, NY: BasicBooks.
- Fama, E.F., and French, K.R. (1988). "Permanent and Temporary Components of Stock Prices". *Journal of Political Economy* 96: 247-273
- Fleming J., Ostdiek B. and Whaley R. E. (1995), "Predicting stock market volatility: a new measure". *Journal of Futures Markets* 15 (1995), pp. 265–302.
- Fouque, Papanicolaou, Sircar, and Solna (2004), "Maturity Cycles in Implied Volatility" *Finance and Stochastics*, Vol. 8, Issue 4, pp 451-477
- Follmer, H., Schweizer, M. (1991) "Hedging of contingent claims under incomplete information, *Applied Stochastic Analysis* (M. H. A. Davis and R. J. Elliott, eds.), Gordon and Breach, New York, 389–414.
- Frittelli M. (2002), "The minimal entropy martingale measure and the valuation problem in incomplete markets, *Mathematical Finance* 10, 39–52.
- Gilboa, I., Schmeidler, D. (2001), "A Theory of Case Based Decisions". Publisher: Cambridge University Press.
- Goll, T., Ruschendorf, L. (2001) "Minimax and minimal distance martingale measures and their relationship to portfolio optimization", *Finance and Stochastics* 5, 557–581
- Greiner, S. P. (2013), "Investment risk and uncertainty: Advanced risk awareness techniques for the intelligent investor". Published by *Wiley Finance*.
- Han, B. (2008), "Investor Sentiment and Option Prices", *The Review of Financial Studies*, 21(1), pp. 387-414.
- Heston S., Nandi, S., 2000. A closed-form GARCH option valuation model. *Review of Financial Studies* 13, 585-625.
- Henderson, V. (2002), "Valuation of claims on nontraded assets using utility maximization", *Mathematical Finance* 12, 351–373
- Henderson, P. W., and Peterson, R. A. (1992), "Mental Accounting and Categorization", *Organizational Behavior and Human Decision Processes*, 51, pp. 92-117.
- Heston S., 1993. "A closed form solution for options with stochastic volatility with application to bond and currency options. *Review of Financial Studies* 6, 327-343.

- Hodges, S. D., Neuberger, A. (1989), "Optimal replication of contingent claims under transaction costs", *The Review of Futures Markets* 8, 222–239
- Hofstadter, D., and Sander, E. (2013), "Surfaces and Essences: Analogy as the fuel and fire of thinking", Published by Basic Books, April.
- Hogarth, R. M., & Einhorn, H. J. (1992). "Order effects in belief updating: The belief-adjustment model". *Cognitive Psychology*, 24.
- Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatilities. *Journal of Finance* 42, 281-300.
- Hume, David (1748), "An Enquiry Concerning Human Understanding". E-version available at <http://creativecommons.org/licenses/by-nc-sa/3.0/au/>
- Kahneman, D., & Frederick, S. (2002). "Representativeness revisited: Attribute substitution in intuitive judgment". In T. Gilovich, D. Griffin, & D. Kahneman (Eds.), *Heuristics and biases* (pp. 49–81). New York: Cambridge University Press.
- Kahneman, D., & Tversky, A. (1982), *Judgment under Uncertainty: Heuristics and Biases*, New York, NY: Cambridge University Press.
- Keynes, John Maynard (1921), "A Treatise on Probability", Publisher: Macmillan And Co.
- Kluger, B., & Wyatt, S. (2004). "Are judgment errors reflected in market prices and allocations? Experimental evidence based on the Monty Hall problem". *Journal of Finance*, pp. 969–997.
- Lakoff, G. (1987), *Women, Fire, and Dangerous Things*, Chicago, IL: The University of Chicago Press.
- Lettau, M., and S. Ludvigson (2010): "Measuring and Modelling Variation in the Risk-Return Trade-off" in *Handbook of Financial Econometrics*, ed. by Y. Ait-Sahalia, and L. P. Hansen, chap. 11, pp. 617-690. Elsevier Science B.V., Amsterdam, North Holland.
- Lustig, H., and A. Verdelhan (2010), "Business Cycle Variation in the Risk-Return Trade-off". Working paper, UCLA.
- Melino A., Turnbull, S., 1990. "Pricing foreign currency options with stochastic volatility". *Journal of Econometrics* 45, 239-265
- Merton, R. C., 1989, "On the Application of the Continuous-Time Theory of Finance to Financial Intermediation and Insurance," *The Geneva Papers on Risk and Insurance* 14, 225-261.
- Miyahara, Y. (2001), "Geometric Lévy Process & MEMM Pricing Model and Related Estimation Problems", *Asia-Pacific Financial Markets* 8, 45–60 (2001)
- Mullainathan, S., Schwartzstein, J., & Shleifer, A. (2008) "Coarse Thinking and Persuasion". *The Quarterly Journal of Economics*, Vol. 123, Issue 2 (05), pp. 577-619.
- Poterba, J.M. and Summers, L.H. (1988). "Mean Reversion in Stock-Prices: Evidence and Implications". *Journal of Financial Economics* 22: 27-59
- Rendleman, R. (2002), *Applied Derivatives: Options, Futures, and Swaps*. Wiley-Blackwell.
- Rennison, G., and Pedersen, N. (2012) "The Volatility Risk Premium", *PIMCO*, September.

- Rockenbach, B. (2004), "The Behavioral Relevance of Mental Accounting for the Pricing of Financial Options". *Journal of Economic Behavior and Organization*, Vol. 53, pp. 513-527.
- Rubinstein, M., 1985. Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active CBOE option classes from August 23, 1976 through August 31, 1978. *Journal of Finance* 40, 455-480.
- Rubinstein, M., 1994. Implied binomial trees. *Journal of Finance* 49, 771-818.
- Soner, H.M., S. Shreve, and J. Cvitanic, (1995), "There is no nontrivial hedging portfolio for option pricing with transaction costs", *Annals of Applied Probability* 5, 327-355.
- Schwert W. G. (1989), "Why does stock market volatility change over time?" *Journal of Finance* 44, 5 (1989), pp. 28-6
- Schwert W. G. (1990), "Stock volatility and the crash of 87". *Review of Financial Studies* 3, 1 (1990), pp. 77-102
- Selten, R.: 1978, 'The Chain-Store Paradox', *Theory and Decision* 9, 127-159.
- Shefrin, H. (2008), "A Behavioral Approach to Asset Pricing", Published by Academic Press, June.
- Shefrin, H. (2010), "Behavioralizing Finance" *Foundations and Trends in Finance* Published by Now Publishers Inc.
- Siddiqi, H. (2009), "Is the Lure of Choice Reflected in Market Prices? Experimental Evidence based on the 4-Door Monty Hall Problem". *Journal of Economic Psychology*, April.
- Siddiqi, H. (2011), "Does Coarse Thinking Matter for Option Pricing? Evidence from an Experiment" *IUP Journal of Behavioral Finance*, Vol. VIII, No.2. pp. 58-69
- Siddiqi, H. (2012), "The Relevance of Thinking by Analogy for Investors' Willingness to Pay: An Experimental Study", *Journal of Economic Psychology*, Vol. 33, Issue 1, pp. 19-29.
- Summers, L. H. (1986). "Does the Stock-Market Rationally Reflect Fundamental Values?" *Journal of Finance* 41: 591-601
- Thaler, R. (1999), "Mental Accounting Matters", *Journal of Behavioral Decision Making*, 12, pp. 183-206.
- Thaler, R. H. "Toward a positive theory of consumer choice" (1980) *Journal of Economic Behavior and Organization*, 1, 39-60
- Wiggins, J., 1987. Option values under stochastic volatility: theory and empirical estimates. *Journal of Financial Economics* 19, 351-372.
- Zaltman, G. (1997), "Rethinking Market Research: Putting People Back In," *Journal of Marketing Research* 34, pp. 424-437.

Appendix A

In the binomial analogy case, the delta-hedged portfolio $S \frac{\Delta C}{\Delta S} - C$ grows at the rate $r + \delta_K$. Divide $[0, T - t]$ in n time periods, and with $n \rightarrow \infty$, the binomial process converges to the geometric Brownian motion. To deduce the analogy based PDE consider:

$$V = S \frac{\partial C}{\partial S} - C$$

$$\Rightarrow dV = dS \frac{\partial C}{\partial S} - dC$$

Where $dS = uSdt + \sigma SdW$ and by Ito's Lemma $dC = \left(uS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW$

$$\Rightarrow (r + \delta_K)Vdt = (uSdt + \sigma SdW) \frac{\partial C}{\partial S} - \left(uS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt - \sigma S \frac{\partial C}{\partial S} dW$$

$$(r + \delta_K)Vdt = - \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt$$

$$\Rightarrow (r + \delta_K) \left(S \frac{\partial C}{\partial S} - C \right) = - \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right)$$

$$(r + \delta_K)C = (r + \delta_K)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \tag{A1}$$

The above is the analogy based PDE.

Appendix B

The analogy based PDE derived in Appendix A can be solved by converting to heat equation and exploiting its solution.

Start by making the following transformation:

$$\tau = \frac{\sigma^2}{2} (T - t)$$

$$x = \ln \frac{S}{K} \Rightarrow S = Ke^x$$

$$C(S, t) = K \cdot c(x, \tau) = K \cdot c\left(\ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T - t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 c}{\partial x^2} - K \cdot \frac{1}{S^2} \frac{\partial c}{\partial x}$$

Plugging the above transformations into (A1) and writing $\tilde{r} = \frac{2(r+\delta_K)}{\sigma^2}$, we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1) \frac{\partial c}{\partial x} - \tilde{r}c \tag{B1}$$

With the boundary condition/initial condition:

$$C(S, T) = \max\{S - K, 0\} \text{ becomes } c(x, 0) = \max\{e^x - 1, 0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau} u + 2\alpha e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau} \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x + \beta \tau} u + e^{\alpha x + \beta \tau} \frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - \tilde{r} - \beta)u + (2\alpha + (\tilde{r} - 1))\frac{\partial u}{\partial x} \quad (B2)$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4}$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad (B3)$$

With the initial condition:

$$u(x_0, 0) = \max\{(e^{(1-\alpha)x_0} - e^{-\alpha x_0}), 0\} = \max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0) dx_0$$

Change variables: $= \frac{x_0-x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0$ iff $x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1 :

$$\frac{\tilde{r} + 1}{2}(x + z\sqrt{2\tau}) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r} + 1)}{2}\right)^2 + \frac{\tilde{r} + 1}{2}x + \tau\frac{(\tilde{r} + 1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z . Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi} - \sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = x/\sqrt{2\pi} + \sqrt{\tau/2}(\tilde{r} + 1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = x/\sqrt{2\pi} + \sqrt{\tau/2}(\tilde{r} - 1)$ and $f = \frac{\tilde{r}-1}{2}x + \tau \frac{(\tilde{r}-1)^2}{4}$

The analogy based European call pricing formula is obtained by recovering original variables:

$$Call = SN(d_1) - Ke^{-(r+\delta_K)(T-t)}N(d_2)$$

Where $d_1 = \frac{\ln(S/K) + (r + \delta_K + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \frac{\ln(\frac{S}{K}) + (r + \delta_K - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$

PREVIOUS WORKING PAPERS IN THE SERIES

FINANCE

- F12_1 Government Safety Net, Stock Market Participation and Asset Prices by Danilo Lopomo Beteto (2012).
- F12_2 Government Induced Bubbles by Danilo Lopomo Beteto (2012).
- F12_3 Government Intervention and Financial Fragility by Danilo Lopomo Beteto (2012).
- F13_1 Analogy Making in Complete and Incomplete Markets: A New Model for Pricing Contingent Claims by Hammad Siddiqi (September, 2013).
- F13_2 Managing Option Trading Risk with Greeks when Analogy Making Matters by Hammad Siddiqi (October, 2012).
- F14_1 The Financial Market Consequences of Growing Awareness: The Case of Implied Volatility Skew by Hammad Siddiqi (January, 2014).
- F14_2 Mental Accounting: A New Behavioral Explanation of Covered Call Performance by Hammad Siddiqi (January, 2014).
- F14_3 The Routes to Chaos in the Bitcoins Market by Hammad Siddiqi (February, 2014).