RISK AND SUSTAINABLE MANAGEMENT GROUP WORKING PAPER SERIES

2013

FINANCE

TITLE:

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Working Paper: F13_1

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Working Paper

September 2013

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Abstract

A key limitation of the Black Scholes model is that it assumes a complete market (claims are replicable with existing assets). We put forward a new option pricing formula that does not require market completeness. The new formula is based on the idea that people rely on, what can be termed, the principle of analogy making while valuing assets. The principle of analogy making says that similar assets should offer the same returns. This is in contrast with the principle of no arbitrage which says that identical assets should offer the same returns. Analogy making is consistent with experimental and anecdotal evidence. I show that the principle of analogy making is a special case of the coarse thinking model of Mullainathan, Schwartzstein, and Shleifer (2008). I then apply the coarse thinking/analogy making model to option pricing and put forward a new option pricing formula. The new formula differs from the Black Scholes formula due to the appearance of a parameter in the formula that captures the risk premium on the underlying. The new formula, called the analogy option pricing formula, provides an explanation for the implied volatility skew puzzle in equity options. The key empirical predictions of the analogy formula are also discussed.

Keywords: *Analogy Making, Incomplete Markets, Implied Volatility, Implied Volatility Skew, Option Prices, Risk Premium*

JEL Classifications: G13; G12

[1] I am grateful to Hersh Shefrin (Santa Clara University), Emanuel Derman (Columbia University), and Don Chance Louisiana State University) for helpful comments and suggestions. All errors and omissions are the sole responsibility of the author.

Analogy Making in Complete and Incomplete Markets: A New Model for Pricing Contingent Claims

People tend to think by analogies and comparisons. Leading cognitive scientists have argued that analogy making forms the core of human cognition and it is the fuel and fire of thinking (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, *"[…] each concept in our mind owes its existence to a long succession of analogies made unconsciously over many years, initially giving birth to the concept and continuing to enrich it over the course of our lifetime. Furthermore, at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known."*
(Hofstadter and Sander (2013), Prologue page1).

According to Hoftsadter and Sander (2013), we engage in analogy making when we spot a link between what we are just experiencing and what we have experienced before. For example, if we see a pencil, we are able to recognize it as such because we have seen similar objects before and such objects are called pencils. Furthermore, spotting of this link is not only useful in assigning correct labels, it also gives us access to a whole repertoire of stored information regarding pencils including how they are used and how much a typical pencil costs. Hofstadter and Sander (2013) argue that analogy making not only allows us to carry out mundane tasks such as using a pencil, toothbrush or an elevator in a hotel but is also the spark behind all of the major discoveries in mathematics and the sciences. They argue that analogy making is responsible for all our thinking, from the most trivial to the most profound. Of course, there is always a danger of making wrong analogies. When a small private plane flew into a building in New York on October 11, 2006, the analogy with the events of September 11, 2001 was irrepressible and the Dow Jones Index fell sharply in response.

The recognition of analogy making as an important decision principle is not new. Hume wrote in 1748, "*From causes which appear similar, we expect similar effects. This is the sum of all our experimental conclusions*". (Hume 1748, Section IV). Similar ideas have been expressed in economic literature by Keynes (1921), Selten (1978), and Cross (1983) among others. To our knowledge, two formal approaches have been proposed to incorporate analogy making into economics: 1) case based decision theory of Gilboa and Schmeidler (2001) and 2) coarse thinking/analogy making model of

Mullainathan, Schwartzstein, and Shleifer (2008). The approach in this paper relates to the model of Mullainathan et al (2008).

Recent experimental evidence suggests that analogy making plays an important role in financial markets. Siddiqi (2012) and Siddiqi (2011) show through controlled laboratory experiments that analogy making strongly influences how much people are willing to pay for call options. These experiments exploit an analogy between in-the-money call options and their underlying stocks and show that in-the-money calls are overpriced due to participants forming an analogy between riskier options and comparatively less risky underlying stocks. The experiments show that people are pricing an in-the-money call based on the principle that similar (not identical) assets should offer the same returns instead of relying on the no-arbitrage principle which says that identical assets should offer the same returns.

In this article, we show that the experimental finding "similar assets should offer same returns" follows from the coarse thinking model of Mullainathan et al (2008). Specifically, it is a special case of their model. We then investigate the theoretical implications of analogy making for option pricing. If investors are pricing a call option in analogy with the underlying stock, then a new option pricing formula (alternative to the Black Scholes formula) is obtained. Unlike the Black Scholes formula, the new formula does not require the assumption of a complete market. The new formula, which we call the analogy option pricing formula, provides an explanation for the implied volatility skew puzzle. Interestingly, the new formula differs only slightly from the Black-Scholes formula by incorporating the risk premium on the underlying in option price. This additional parameter is sufficient to explain implied volatility. We also provide testable predictions of the analogy option pricing formula.

This paper adds to the literature in several ways. 1) In an interesting paper, Derman (2002) writes, "*If options prices are generated by a Black–Scholes equation whose rate is greater than the true riskless rate, and if these options prices are then used to produce implied volatilities via the Black–Scholes equation with a truly riskless rate, it is not hard to check that the resultant implied volatilities will produce a negative volatility skew.*" (Derman (2002) page 295)

This paper provides a reason for the above mentioned effect.[2] The analogy formula is exactly identical to the Black Scholes formula apart from replacing the risk free rate with the return on the underlying stock. 2) We show how a particular decision rule, called analogy making, leads to a new option pricing formula and explains the implied volatility puzzle. In particular, we show that the

[2] I am grateful to Emanuel Derman for pointing this out.

implied volatility skew is generated if actual price dynamics are determined according to the analogy formula and the Black-Scholes formula is used to back-out implied volatility. Our approach is an application of the coarse thinking model of Mullainathan et al (2008). Our approach is also broadly consistent with Shefrin (2008) who provides a systematic treatment of how behavioral assumptions impact the pricing kernel at the heart of modern asset pricing theory. However, the treatment here differs from Shefrin (2008) as we focus on one particular decision rule and explore its implications. 3) It is well known that in an incomplete market there is no unique no-arbitrage price; rather a wide interval of arbitrage-free prices is obtained as the martingale measure is not unique. Which price to pick then? Two approaches have been developed to search for solutions in an incomplete market. One is to pick a specific martingale measure according to some optimal criterion. See Follmer and Schweizer (1991), Miyahara (2001), Fritelli (2002), Bellini and Fritelli (2002), and Goll and Ruschendorf (2001) among others. The other approach is utility based option pricing. See Hodges and Neuberger (1989), Davis (1997), and Henderson (2002) for early treatment. The approach in this paper relates to the former as it specifies a mechanism for picking a specific martingale measure. 4) We provide a number of testable predictions of the model and summarize existing evidence. The existing evidence is strongly in favor of the analogy approach. 5) Our approach relates to Bollen and Whaley (2004). They argue that, in the presence of limits to arbitrage, net demand pressure could determine the level and the slope of the implied volatility curve. In our approach, the source of demand pressure behind the skew is analogies that investors make between in-the-money calls and the underlying stocks. Such analogies lead them to consider in-the-money calls as replacements of the underlying stocks.[3] 6) Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total risk, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. In the analogy option pricing model, higher risk premium for a given level of total volatility generates this result. As risk premium is related to systematic risk, this prediction of the analogy model is quite intriguing. 7) Our approach is an example of behavioralization of finance. Shefrin (2010) argues that finance is in the midst of a paradigm shift,

[3] Option traders and investment professionals often advise people to buy in-the-money calls rather than the underlying stocks. As illustrative examples, see the following:
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/,
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

from a neoclassical based framework to a psychologically based framework. Behavioralizing finance is the process of replacing neoclassical assumptions with behavioral counterparts while maintaining mathematical rigor.

In general, pricing models that have been proposed to explain the implied volatility skew can be classified into three broad categories: 1) Stochastic volatility and GARCH models (Heston and Nandi (2000), Duan (1995), Heston (1993), Melino and Turnbull (1990), Wiggins (1987), and Hull and White (1987)). 2) Models with jumps in the underlying price process (Amin (1993), Ball and Torous (1985)). 3) Models with stochastic volatility as well as random jumps. See Bakshi, Cao, and Chen (1997) for a discussion of their empirical performance (mixed). Most of these models modify the price process of the underlying. Hence, the focus of these models is on finding the right distributional assumptions that could explain the implied volatility puzzles. Our approach differs from them fundamentally as we do not modify the underlying price process. That is, as in the Black Scholes model, we continue to assume that the underlying's stochastic process is a constant coefficient geometric Brownian motion. To our knowledge, ours is the only model that explains the implied volatility skew without modifying the geometric Brownian motion assumption of the Black-Scholes model. Hence, it is not necessary to modify the assumption of geometric Brownian motion to generate implied volatility skew.

This paper is organized as follows. Section 1 illustrates the difference between the principle of no arbitrage and the principle of analogy making through a simple example in a complete market context. Section 2 illustrates analogy making in an incomplete market and shows that the principle of analogy making picks out a specific martingale measure from the set of allowable martingale measures. Section 3 shows that the principle of analogy making is a special case of the coarse thinking model of Mullainathan et al (2008). Section 4 explores the implications of analogy making in a one period binomial model, which is the simplest case of a complete market. Section 5 does the same for the simplest case of an incomplete market: the trinomial model. Section 6 puts forward the analogy option pricing formula that does not require the assumption of a complete market. Section 7 shows if prices are determined in accordance with the analogy formula and the Black Scholes formula is used to infer implied volatilities then the implied volatility skew is observed. Section 8 puts forward the key empirical predictions of the model. Section 9 concludes.

1. Analogy Making: A Complete Market Example

Consider an investor in a two state-two asset complete market world. The investor has initially put his money in the two assets: A stock (S) and a risk free bond (B). The stock has a price of $140 today. In the next period, the stock could either go up to $200 (the red state) or go down to $90 (the blue state). Each state has a 50% chance of occurring. The bond costs $100 today and it also pays $100 in the next period implying a risk free rate of zero. Suppose a new asset "A" is introduced to him. The asset "A" pays $140 in the red state and $30 in the blue state. How much should he be willing to pay for it?

Conventional finance theory provides an answer by appealing to the principle of no-arbitrage: *identical assets should offer the same returns.* An asset identical to "A" is a portfolio consisting of a long position in S and a short position in 0.60 of B. In the red state, S pays $200 and one has to pay $60 due to shorting 0.60 of B resulting in a net payoff of $140. In the blue state, S pays $90 and one has to pay $60 on account of shorting 0.60 of B resulting in a net payoff of $30. That is, payoffs from S-0.60B are identical to payoffs from "A". Hence, according to the no-arbitrage principle, "A" should be priced in such a way that its expected return is equal to the expected return from (S-0.60B). It follows that the no-arbitrage price for "A" is $80.

In practice, constructing a portfolio that replicates "A" is no easy task. When simple tasks such as the one described above are presented to participants in a series of experiments, they seem to rely on analogy-making to figure out their willingness to pay. See Siddiqi (2012) and Siddiqi (2011). So, instead of trying to construct a replicating portfolio which is identical to asset "A", people find an actual asset similar to "A" and price "A" in analogy with that asset. That is, they rely on the principle of analogy: *similar assets should offer the same returns* rather than on the principle of no-arbitrage: *identical assets should offer the same returns.*

Asset "A" is similar to asset S. It pays more when asset S pays more and it pays less when asset S pays less. Expected return from S is 1.0357 $\left(\frac{0.5\times200+0.5\times90}{140}\right)$. According to the principle of analogy, A's price should be such that it offers the same expected return as S. That is, the right price for A is $82.07.

In the above example, there is a gap of $2.07 between the no-arbitrage price and the analogy price. Rational investors should short "A" and buy "S-0.60B". However, if we introduce a small transaction cost of 1%, then the total transaction cost of the proposed scheme exceeds $2.07, preventing arbitrage. The transaction cost of shorting "A" is \$0.8207 whereas the transaction cost of buying "S-0.60B" is \$1.6 so the total transaction cost is \$2.4207. Hence, in principle, the deviation between the no-arbitrage price and the analogy price may not be corrected due to transaction costs even if we assume a complete market. It is also interesting to note that asset "A" is equivalent to a call option on "S" with a strike price of 60.

Even though the example discussed above is a complete market example in which asset "A" can be replicated by using other assets, it is important to realize that the idea of analogy making does not require complete markets. It works equally well, if not better, in incomplete markets. For example, one can easily add another state, say Green, with a payoff that makes replication impossible. Consequently, we do not get a unique rational price as a benchmark; however, a unique analogy price is still obtained. Market incompleteness determines fairly wide bounds for the rational price so, in principle, analogy making may pick out one price within the allowed rational bounds.

It is also important to realize that the notion of similarity is an inherently subjective notion. "Spotting of a link between what one is experiencing now and what one has experienced before" depends on the kinds of experiences people have had previously. Of course, different people have different repertoire of experiences. Two people landing in Brisbane for the first time may compare it to different cities they have been to earlier. One may find it similar to Santa Clara due to similarity in architecture. Another may say it is similar to San Francisco due to hills. Analogously, when an asset that a person is not familiar with is presented to her, she may compare it to a similar asset that she has been exposed to earlier. What constitutes "similar" varies from person to person and depends on the prior experiences of that person. However, sometimes the similarity is naturally pointed out in the context of the decision problem. This is the case with call options. A call option is a right to buy the underlying, so its payoff depends on the payoffs from the underlying. Clearly, the similarity between a call option and its underlying stock is pointed out in the decision context. This similarity is even stronger for an in-the-money call that moves nearly dollar-for-dollar with the underlying. In other words, in the case of call options, the context persuades an investor to consider the similarity with the underlying. That's why investment professionals with decades of experience frequently advise clients to replace stocks in their portfolios with call options (in-the-money calls) as they consider them surrogates for the underlying stocks.[4] In fact, they use a call option's *moneyness* as a

[4] As few examples, see: http://www.triplescreenmethod.com/TradersCorner/TC052705.asp
http://www.streetdirectory.com/travel_guide/36660/investment/option_trading_tip___buy_deep_in_the_money_options.html

measure of similarity with more in-the-money call options considered more similar to the underlying.

Inspired by market professionals, for call options, one can use *moneyness* as a measure of *similarity*. In the example discussed above, *moneyness* $= {}^{K}\!/_{S} = {}^{60}\!/_{140} = 0.43$ where K is the strike price which is 60 in this case and S is the stock price. For a given price of the underlying, lower values of *moneyness* imply a greater degree of *similarity* (as K is lower) with the greatest similarity achieved when *moneyness* is 0 (when $K = 0$), which is the case of identical assets.

2. Analogy Making: An Incomplete Market Example

Consider the simplest incomplete market in which there are two assets and three states. Each state is equally likely to occur. Asset "S" has a price of 120 today and the risk free asset "B" has a price of 100 today. The state-wise payoffs are summarized in table 1.

Table 1

Asset Type	**Price**	**Red**	**Blue**	**Green**
S	120	200	90	120
B	100	110	110	110

Consider the following claim: Red state payoff is 140; Blue state payoff is 30; and Green state payoff is 100. This claim cannot be replicated with S and B. Hence, there is no unique no-arbitrage price. However, an arbitrage free interval can be specified: $65.5 < arbitrage\ free\ price < 101.8$.

Applying the principle of analogy making picks out the following price from the arbitrage free interval: 79.02. It can be shown that this price corresponds to the following martingale measure: (0.280044, 0.346785, 0.373171). Hence, in this example, analogy making picks out a specific martingale measure from the set of allowable martingale measures. Consequently, it can be considered a selection mechanism.

Next, we show that the principle of analogy making can be considered a special case of the coarse thinking model of Mullainathan, Schwartzstein, and Shleifer (2008).

3. Coarse thinking/Analogy Making

According to the coarse thinking model of Mullainathan et al (2008), a coarse thinker co-categorizes situations that he considers similar or analogous and assessment of attributes in a given situation is affected by the assessment of corresponding attributes in co-categorized situations. This is called *transference*. To understand this, we re-produce equation (7) from their paper below (with slight change in notation):

$$E^C[q|I,s=0]=E[q|I,s=0]w(s=0|C)+E[q|I,s=1]w(s=1|C) \qquad (I)$$

In the above equation, the attribute of concern is quality which is denoted by q. The symbol C indicates category/category based thinking/coarse thinking. I indicates information (both public and private). The purpose is to form an expectation about quality of an item which is denoted by 's=0'. However, the item is co-categorized with another item which is denoted by 's=1'. The quality of the item 's=1' influences the perception of the quality of the item 's=0' in the mind of a coarse thinker according to their respective weights in the category: w(s=0|C) and w(s=1|C). The weights sum to one. Such influence across items in the same category is termed *transference*.

The connection of the above equation with the example given in section 1 is straightforward. Specifically, in the example, one is interested in figuring out what expected return to demand from the new asset "A". The new asset "A" is similar to the old/familiar asset "S". Hence, "A" gets co-categorized with "S". We denote the return on an asset by $q \in Q$, where Q is some subset of $\Re$ (the set of real numbers). In figuring out the return to expect from the new asset, an analogy maker faces two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s=0$, "return demanded on the new asset" is the attribute of interest and in $s=1$, "actual return available on the old/familiar asset" is the attribute of interest. The analogy maker has access to all the information described above. We denote this public information by I. The principle of analogy says *similar assets should offer the same returns.* It follows:

$$E^C[q|I,s=0]=E[q|I,s=1] \qquad (II)$$

Equation (II) follows from equation (I) if w(s=1|C)=1.

Hence, the principle of analogy as formulated in this article is a special case of the coarse thinking model. It corresponds to the case where the co-categorized situation has the strongest influence on the assessment of the attribute of concern.

Next, we apply the coarse thinking model/analogy making model of Mullainathan et al (2008) to the simplest case of complete markets: the binomial model. The fundamental insights developed in the binomial case carry over to the general case.

4. Analogy Making: The Binomial Case

Consider a simple two state world. The equally likely states are Red, and Blue. There is a stock with payoffs $X_1, and\ X_2$ corresponding to states Red, and Blue respectively. The state realization takes place at time T. The current time is time t. We denote the risk free discount rate by r. The current price of the stock is S. There is another asset, which is a call option on the stock. By definition, the payoffs from the call option in the two states are:

$$C_1 = max\{(X_1 - K), 0\}\,, C_2 = max\{(X_2 - K), 0\} \tag{1}$$

Where K is the striking price, and $C_1, and\ C_2,$ are the payoffs from the call option corresponding to Red, and Blue states respectively.

As can be seen, the payoffs in the two states depend on the payoffs from the stock in corresponding states. Furthermore, by appropriately changing the striking price, the call option can be made more or less similar to the underlying stock with the similarity becoming exact as K approaches zero (all payoffs are constrained to be non-negative). As our focus is on in-the-money call options, we assume:

$X_1 - K > 0, and\ X_2 - K > 0.$

How much is an analogy maker willing to pay for this call option?

An analogy maker co-categorizes this call option with the underlying and values it in *transference* with the underlying stock. In other words, an analogy maker relies on the principle of analogy: *similar* *assets should offer the same return*. In contrast, a rational investor relies on the principle of no-arbitrage: *identical assets should offer the same return.*

We denote the return on an asset by $q \in Q$, where Q is some subset of $\Re$ (the set of real numbers). In calculating the return of the call option, an analogy maker faces two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s = 0$, "return demanded on the call option" is the attribute of interest and in $s = 1$, "actual return available on the underlying stock" is the attribute of interest. The analogy maker has access to all the information described above. We denote this public information by I.

The actual expected return available on the underlying stock is given by,

$$E[q|I, s = 1] = \frac{(X_1 - S) + (X_2 - S)}{2S} \tag{2}$$

For the analogy maker, the expected return demanded on the call option is:

$$\begin{aligned} E[q|I, s = 0] &= E[q|I, s = 1] \\ &= \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \end{aligned} \tag{3}$$

So, the analogy maker infers the price of the call option, P_c, from:

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \tag{4}$$

It follows,

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$=> P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \tag{5}$$

We know,

$$S = e^{-(r+\delta)(T-t)} \times \frac{X_1 + X_2}{2} \tag{6}$$

where δ is the risk premium on the underlying.

Substituting (6) into (5):

$$P_c = S - Ke^{-(r+\delta)(T-t)} \quad (7)$$

The above equation is the one period analogy option pricing formula for in-the-money binomial case.

The rational price P_r is (from the principle of no-arbitrage):

$$P_r = S - Ke^{-r(T-t)} \quad (8)$$

If limits to arbitrage prevent rational arbitrageurs from making riskless profits at the expense of analogy makers, both types will survive in the market. If α is the weight of rational investors in the market price and $(1-\alpha)$ is the weight of analogy makers, then the market price becomes:[5]

$$P_c^M = \alpha\left(S - Ke^{-r(T-t)}\right) + (1-\alpha)\left(S - Ke^{-(r+\delta)(T-t)}\right) \quad (9)$$

Proposition 1 *The price of a call option in the presence of analogy makers* $(\alpha < 1)$ *is always larger than the price in the absence of analogy makers* $(\boldsymbol{\alpha = 1})$ *as long as the underlying stock price reflects a positive risk premium. Specifically, the difference between the two prices is* $(\mathbf{1}-\boldsymbol{\alpha})\left(\boldsymbol{K}\boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})} - \boldsymbol{K}\boldsymbol{e}^{-(\boldsymbol{r}+\boldsymbol{\delta})(\boldsymbol{T}-\boldsymbol{t})}\right)$ *where δ is the risk premium reflected in the price of the underlying stock.*

Proof.

Subtracting equation (8) from equation (9) yields the desired expression which is greater than zero as long as $\delta > 0$.

■

[5] In general, alpha weighted average is obtained if we assume log utilities.

Proposition 2 shows that the presence of analogy makers does not change the Sharpe-ratio of options. This shows that the mispricing of options with respect to the underlying due to the presence of analogy makers is not reflected in the Sharpe-ratio as the change in expected excess return is offset by the change in standard deviation.

Proposition 2 *The Sharpe-ratio of a call option remains unchanged regardless of whether the analogy makers are present or not. Specifically, the Sharpe-ratio remains equal to the Sharpe-ratio of the underlying regardless of the presence of analogy makers.*

Proof.

Initially, assume that analogy makers are not present. Let x be the number of units of the underlying stock needed and let B be the dollar amount invested in a risk-free bond to create a portfolio that replicates the call option. It follows,

$$P_c = Sx + B \qquad (I)$$

$$X_1 - K = xX_1 + B(1+r) \qquad (II)$$

Substitute B from (I) into (II) and re-arrange to get:

$$X_1 - K - P_c = x(X_1 - S) - xSr + P_c r$$

$$\Rightarrow \Delta C = x\Delta S - xSr + P_c r$$

$$\Rightarrow \Delta C = xS\left(\frac{\Delta S}{S} - r\right) + P_c r$$

$$\Rightarrow \frac{\Delta C}{P_c} = x\frac{S}{P_c}\left(\frac{\Delta S}{S} - r\right) + r$$

$$\Rightarrow R_c = \Omega(R_s - r) + r$$

Where Ω is the elasticity of option's price with respect to the stock price, R_c and R_s are returns on call and stock respectively. It follows,

$$E[R_c] = \Omega(E[R_s - r]) + r \qquad (III)$$

$$Var[R_c] = \Omega^2 Var[R_s]$$

$$\Rightarrow \sigma_c^2 = \Omega^2 \sigma_s^2 \qquad (IV)$$

$$Sharpe - ratio\ of\ Call\ without\ analogy\ makers = \frac{E[R_c] - r}{\sigma_c}$$

$$= \frac{\Omega(E[R_s] - r) + r - r}{\Omega \sigma_s} = \frac{E[R_s] - r}{\sigma_s} = Sharpe - ratio\ of\ the\ underlying\ stock$$

If analogy makers are also present, it follows,

$$E[R_c] = \alpha\{\Omega(E[R_s - r]) + r\} + (1 - \alpha)E[R_s] \qquad (V)$$

$$\sigma_c^2 = \{\alpha\Omega + (1 - \alpha)\}^2 \sigma_s^2 \qquad (VI)$$

$$\mathrm{Sharpe - ratio\ of\ Call\ with\ analogy\ makers} = \frac{\alpha\{\Omega(E[R_s - r] + r)\} + (1 - \alpha)E[R_s] - r}{\{\alpha\Omega + (1 - \alpha)\}\sigma_s}$$

$$= \frac{\{\alpha\Omega + (1 - \alpha)\}E[R_s] - r\{\alpha\Omega + (1 - \alpha)\}}{\{\alpha\Omega + (1 - \alpha)\}\sigma_s} = \frac{E[R_s] - r}{\sigma_s} = Sharpe - ratio\ of\ the\ underlying\ stock$$

■

Proposition 2 shows that the mispricing caused by the presence of analogy makers does not change the Sharpe-ratio. It remains equal to the Sharpe-ratio of the underlying. Hence, the fact that the empirical Sharpe-ratios of call options and underlying stocks do not differ cannot be used to argue that there is no mispricing in options with respect to the underlying.

Proposition 3 shows the condition under which rational arbitrageurs cannot make arbitrage profits at the expense of analogy makers. Consequently, both types may co-exist in the market.

Proposition 3 *Analogy makers cannot be arbitraged out of the market if*

$(\boldsymbol{1}-\boldsymbol{\alpha})\{\boldsymbol{K}\boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})}-\boldsymbol{K}\boldsymbol{e}^{-(\boldsymbol{r}+\boldsymbol{\delta})(\boldsymbol{T}-\boldsymbol{t})}\} < c$ ***where c is the transaction cost involved in the arbitrage scheme and δ>0***

Proof.

The presence of analogy makers increases the price of an in-the-money call option beyond its rational price. A rational arbitrageur interested in profiting from this situation should do the following: Write a call option and create a replicating portfolio. If there are no transaction costs involved then he would pocket the difference between the rational price and the market price without creating any liability for him when the option expires. As proposition 1 shows, the difference is $(1-\alpha)\{\mathrm{K}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}-\mathrm{K}\mathrm{e}^{-(\mathrm{r}+\delta)(\mathrm{T}-\mathrm{t})}\}$. However, if there are transactions costs involved then he would follow the strategy only if the benefit is greater than the cost. Otherwise, arbitrage profits cannot be made.

■

Analogy makers overprice a call option. When such overpriced calls are added to portfolios then the dynamics of such portfolios would be different from the dynamics without overpricing. Proposition 4 considers the case of covered call writing and shows that the two portfolios grow with different rates with time.

Proposition 4 ***If analogy makers set the price of an in-the-money call option then the covered call writing position (long stock+short call) grows in value at the rate of*** $\mathbf{r}+\boldsymbol{\delta}$***. If rational investors set the price of an in-the-money call option then the covered call writing position grows in value at the risk free rate*** $\mathbf{r}$***.***

Proof.

Re-arranging equation (8):

$Ke^{-r(T-t)} = S - P_r$

The right hand side of the above equation is the covered call writing position with rational pricing. Hence, it follows that the covered call portfolio grows in value at the risk free rate with time if investors are rational.

For analogy makers, re-arrange equation (7):

$$Ke^{-(r+\delta)(T-t)} = S - P_c$$

The right hand side of the above equation is the covered call writing position when analogy makers price the call option. As the left hand side shows, this portfolio grows at a rate of $r + \delta$ with time.

■

Corollary 4.1 *Proposition 4 extends to the case when the portfolio is* $\boldsymbol{S \cdot \frac{\Delta C}{\Delta S} - C}$ *instead of covered call writing where C is the price of a call option.*

Proof. Follows from realizing that for in-the-money binomial case, $\frac{\Delta C}{\Delta S} = 1$.

Corollary 4.2 *Suppose the call option is out of the money in one state, proposition 4 still extends to the case when the portfolio is* $\boldsymbol{S \cdot \frac{\Delta C}{\Delta S} - C}$ *where C is the price of a call option.*

Proof. Analogy Case: First consider the case where the option is out of the money in the blue state. It follows that $\frac{\Delta C}{\Delta S} = \frac{X_1}{X_1+X_2}$. So, $S \cdot \frac{\Delta C}{\Delta S} - C = 2Ke^{-(r+\delta)(T-t)}$ which grows to $2K$ on expiry. If the option is out-of-the-money in the red state, then $\frac{\Delta C}{\Delta S} = \frac{X_2}{X_1+X_2}$, so $S \cdot \frac{\Delta C}{\Delta S} - C = 2Ke^{-(r+\delta)(T-t)}$ which grows to $2K$on expiry.

No-arbitrage Case: If the option is out-of-the-money in the blue state, then $\frac{\Delta C}{\Delta S} = \frac{X_1-K}{X_1-X_2}$. It follows that $S \cdot \frac{\Delta C}{\Delta S} - C = X_2 \frac{X_1-K}{X_1-X_2} e^{-(r)(T-t)}$ which grows at the rate r. Similarly, if the option is out-of-the money in the red state then $\frac{\Delta C}{\Delta S} = \frac{X_2-K}{X_2-X_1}$. Hence, $S \cdot \frac{\Delta C}{\Delta S} - C = X_1 \frac{X_2-K}{X_2-X_1} e^{-(r)(T-t)}$ which grows at the rate r.

■

Proposition 4 and its corollaries show that the portfolio $\mathrm{S} \cdot \frac{\Delta \mathrm{C}}{\Delta \mathrm{S}} - \mathrm{C}$ grows at different rates under analogy vs. no arbitrage pricing. This fact is crucial in deriving the complete market analogy option pricing formula when the underlying follows geometric Brownian motion.

5. Analogy Making: The Trinomial Case

The simplest example of an incomplete market is a trinomial model with three states and two assets. In such a market, in general, it is not possible to replicate a given claim by using existing assets. The equally likely states are Red, Blue, and Green. There is a stock with payoffs $X_1, X_2, and\ X_3$ corresponding to states Red, Blue, and Green respectively. The state realization takes place at time T. The current time is time t. The second asset is a risk free asset with the risk free rate of return given by r. The current price of the stock is S.

Suppose, a new asset is introduced, which is a call option on the stock. By definition, the payoffs from the call option in the three states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\},$$

$$and\ C_3 = max\{(X_3 - K), 0\} \tag{10}$$

Where K is the striking price, and $C_1, C_2, and\ C_3$ are the payoffs from the call option corresponding to Red, Blue, and Green states respectively. To ensure that this claim is non-replicable, we need one state payoff from the call option to be 0. Without loss of generality, we assume the following:

$$C_1 = X_1 - K, C_2 = X_2 - K, and\ C_3 = 0$$

Proposition 5 shows the rate at which the portfolio $\mathrm{S} \cdot \frac{\Delta \mathrm{C}}{\Delta \mathrm{S}} - \mathrm{C}$ grows under analogy making.

Proposition 5 ***If analogy makers set the price of the call option then the portfolio $S \cdot \frac{\Delta C}{\Delta S} - C$ grows at the rate $\mathbf{r} + \boldsymbol{\delta}$ where C is the price of a call option.***

Proof. First notice that $\frac{\Delta C}{\Delta S} = \frac{X_1 + X_2}{X_1 + X_2 + X_2}$. It follows that $\mathrm{S} \cdot \frac{\Delta \mathrm{C}}{\Delta \mathrm{S}} - \mathrm{C} = 6\mathrm{K}e^{-(r+\delta)(T-t)}$ which grows at the rate $r + \delta$. Hence proved ■

It is interesting to note that it does not matter whether we assume a complete market binomial process or an incomplete market trinomial process, the rate of growth of the delta hedging portfolio ($S \cdot \frac{\Delta C}{\Delta S} - C$) remains the same. As both the binomial process and the trinomial process converge to the geometric Brownian motion in the limit of small time steps, it is easy to see that the resulting analogy formula takes the same form regardless of market completeness.

6. The Option Pricing Formula

In this section, we derive a new option pricing formula by allowing the underlying to follow a geometric Brownian motion instead of the restrictive binomial or trinomial processes assumed in the previous sections. It is well known that the Brownian motion is the limiting case of both the binomial and trinomial processes. By exploring the implications of analogy making in the binomial and trinomial cases, the previous section develops intuition which carries over to the general case discussed here. Two key results from the previous sections stand out in this respect. Firstly, a new parameter, δ, which is the risk premium on the underlying stock, appears in the analogy price with the analogy price being larger than the rational price as long as the risk premium is positive in the binomial case. Of course, there is no unique rational price in the trinomial case and the analogy price may be among the no-arbitrage prices. Secondly, in both the binomial and trinomial cases, the rate of growth of the delta hedging portfolio ($S \cdot \frac{\Delta C}{\Delta S} - C$) remains the same and is equal to $r + \delta$. As both the trinomial and binomial processes converge to geometric Brownian motion, it does not matter whether we assume market completeness (as in the binomial process) or incompleteness (as in the trinomial process). The formula remains the same.

Proposition 6 considers the case when the underlying follows geometric Brownian motion and derives the analogy option pricing partial differential equation (PDE) for European call options.

Proposition 6 *If the underlying follows geometric Brownian motion, the Analogy Option Pricing Partial Differential Equation (PDE) is*

$$(\boldsymbol{r} + \boldsymbol{\delta})\boldsymbol{C} = \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{t}} + \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{S}}(\boldsymbol{r} + \boldsymbol{\delta})\boldsymbol{S} + \frac{\partial^2 \boldsymbol{C}}{\partial \boldsymbol{S}^2}\frac{\boldsymbol{\sigma}^2 \boldsymbol{S}^2}{\mathbf{2}}$$

Proof.

See Appendix A ■

The analogy option pricing PDE can be solved by transforming it into the heat equation. Proposition 7 shows the resulting call option pricing formula for European options.

Proposition 7 *The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is* $\boldsymbol{C = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)}$ *where* $\boldsymbol{d_1 = \frac{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}$ *and* $\boldsymbol{d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$

Proof.

See Appendix B.

■

Corollary 7.1 *The formula for the analogy based price of a European put option is* $\boldsymbol{Ke^{-(r+\delta)(T-t)}N(-d_2) - SN(-d_1)}$

The analogy option pricing formula is different from the Black-Scholes formula due to the appearance of risk premium on the underlying in the analogy formula. It suggests that the risk premium on the underlying stock does matter for option pricing. The analogy formula is derived by keeping all the assumptions behind the Black-Scholes formula except one: in the case of complete markets, the assumption of no-arbitrage pricing is dropped, and in the case of incomplete markets, of course, the assumption of market completeness is dropped.

7. The Implied Volatility Skew

All the variables in the Black Scholes formula are directly observable except for the standard deviation of the underlying's returns. So, by plugging in the values of observables, the value of standard deviation can be inferred from market prices. This is called implied volatility. If the Black Scholes formula is correct, then the implied volatility values from options that are equivalent except for the strike prices should be equal. However, in practice, for equity index options, a skew is observed in which in-the-money call options' (out-of-the money puts) implied volatilities are higher than the implied volatilities from at-the-money and out-of-the-money call options (in-the-money puts).

The analogy approach developed here provides an explanation for the skew. If the analogy formula is correct, and the Black Scholes model is used to infer implied volatility then skew arises as table 1 shows.

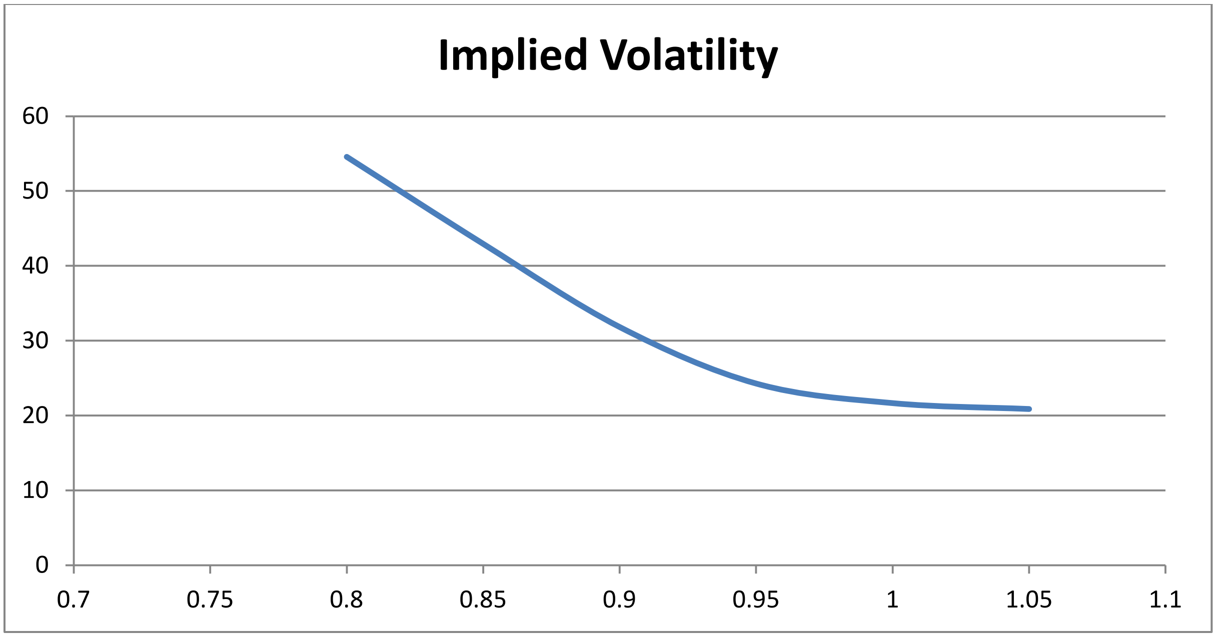
Table 1

Implied Volatility Skew

Underlying's Price=100, Volatility=20%, Risk Premium on the Underlying=5%, Time to Expiry=0.06 year

K	**Black Scholes Price**	**Analogy Price**	**Difference**	**Implied Volatility**
105	0.5072	0.5672	0.06	20.87
100	2.160753	2.326171	0.165417	21.6570
95	5.644475	5.901344	0.25687	24.2740
90	10.30903	10.58699	0.277961	31.8250
85	15.26798	15.53439	0.266419	42.9400
80	20.25166	20.50253	0.250866	54.5700

As table 1 shows, implied volatility skew is seen if the analogy formula is correct, and the Black Scholes formula is used to infer implied volatility. Notice that in the example considered, difference between the Black Scholes price and the analogy price is quite small even when implied volatility gets more than double the value of actual volatility.



K/S

Figure 1

Figure 1 is the graphical illustration of table 1. It is striking to observe from table 1 and figure 1 that the implied volatility skew is quite steep even when the price difference between the Black Scholes price and the analogy price is small. In the next section, we outline a number of key empirical predictions that follow from the analogy making model.

8. Key Predictions of the Analogy Model

Prediction#1 ***After controlling for the underlying asset's total volatility, a higher amount of risk premium on the underlying leads to a higher level of implied volatility and a steeper slope of the implied volatility curve.***

Risk premium on the underlying plays a key role in analogy option pricing formula. Higher the level of risk premium for a given level of volatility; higher is the extent of overpricing. So, higher risk premium leads to higher implied volatility levels at all values of moneyness. Figure 2 illustrates this. In the figure, implied volatility skews for two different values of risk premia are plotted. Other parameters are the same as in table 1. Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total risk, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. As risk premium is related to systematic risk, the prediction of the analogy model is quite intriguing.

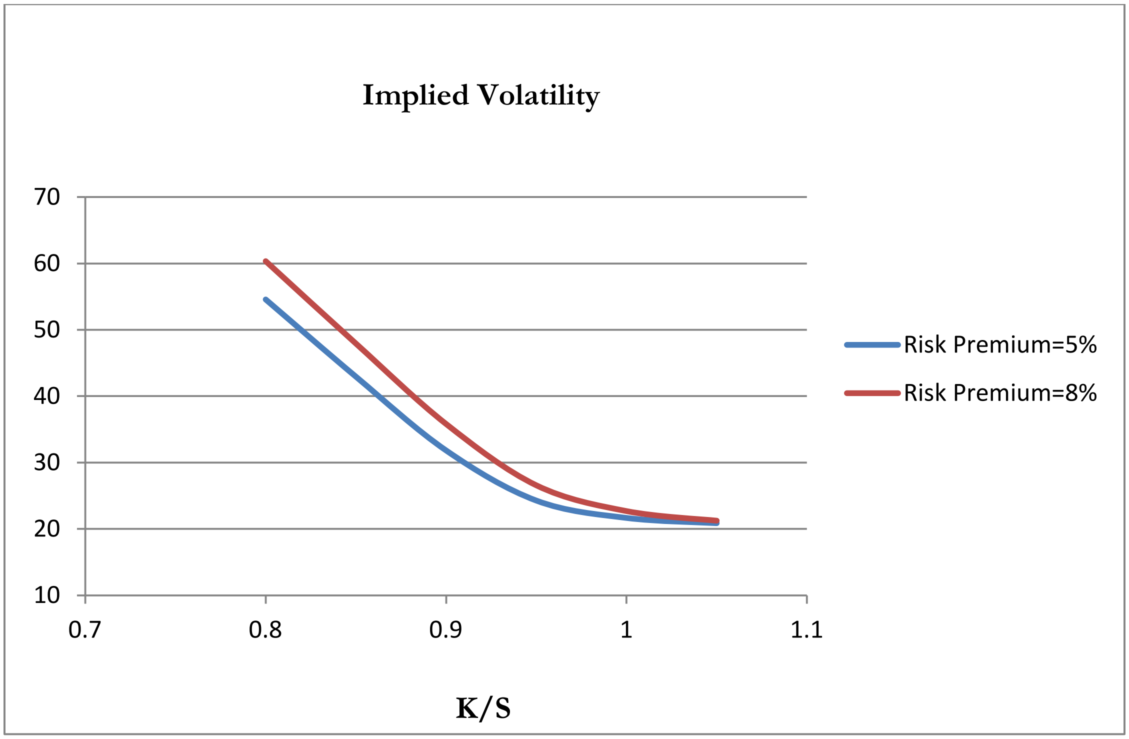


Figure 2

Prediction#2 ***Implied volatility should typically be higher than realized/historical volatility***

It follows directly from the analogy formula that as long as the risk premium on the underlying is positive, implied volatility should be higher than actual volatility. Anecdotal evidence is strongly in favor of this prediction. Rennison and Pederson (2012) calculate implied volatilities from at-the-money options in 14 different options markets over a period ranging from 1994 to 2012. They show that implied volatilities are typically higher than realized volatilities.

Prediction#3 ***Implied volatility curve should flatten out with expiry***

Figure 3 plots implied volatility curves for two different expiries. All other parameters are the same as in table 1. It is clear from the figure that as expiry increases, the implied volatility curve flattens out.

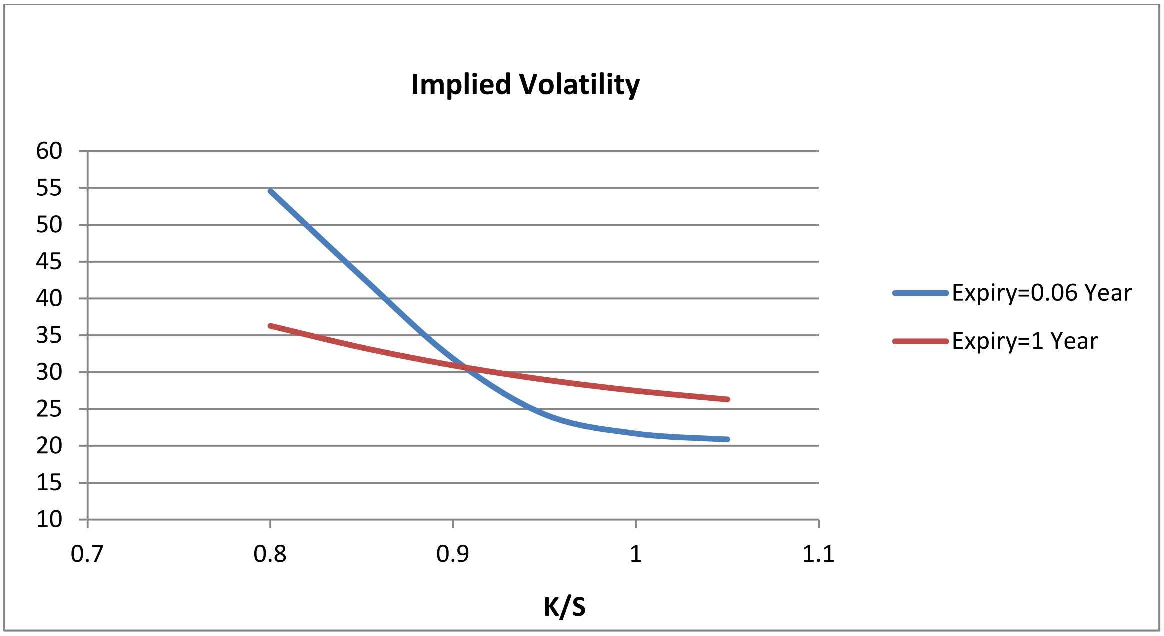


Figure 3

Empirically, implied volatility curve typically flattens out with expiry (see Greiner (2013) as one example). Hence, this match between a key prediction of the analogy model and empirical evidence is quite intriguing.

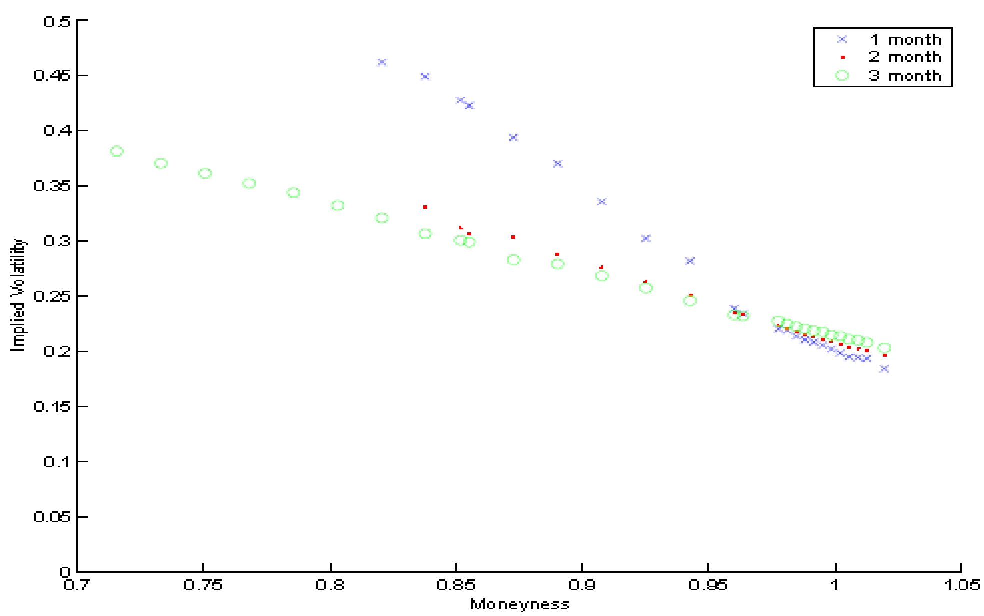


Figure 4 *Implied volatility as a function of moneyness on January 12, 2000, for options with at least two days and at most three months to expiry.*

As an illustration of the fact that implied volatility curve flattens with expiry, figure 4 is a reproduction of a chart from Fouque, Papanicolaou, Sircar, and Solna (2004) (figure 2 from their paper). It plots implied volatilities from options with at least two days and at most three months to expiry. The flattening is clearly seen.

8. Conclusion

Analogy making appears to be the key to the way we think. In this article, we investigate the implications of analogy making for option pricing. We put forward a new option pricing formula that we call the analogy option pricing formula. The new formula differs with the Black Scholes formula due to the introduction of a new parameter capturing risk premium on the underlying stock. The new formula provides an explanation for the implied volatility skew puzzle. Three testable predictions of the model are also discussed.

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Appendix A

In the binomial model, the portfolio $Sx - C$ grows at the risk free rate r if the no-arbitrage principle is followed. However, if the principle of analogy making is followed then the same portfolio grows at the rate $r + \delta$. As before, x is the number of units of the underlying stock in the replicating portfolio, and δ is the risk premium on the underlying stock. Divide $[0, T - t]$ in n time periods, and with $n \to \infty$, the binomial model converges to geometric Brownian motion. To deduce the analogy based PDE consider:

$V = Sx - C$

$\Rightarrow dV = dSx - dC$

Where $dS = uSdt + \sigma SdW$ and by Ito's Lemma $dC = \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW$

$\Rightarrow (r + \delta)Vdt = (uSdt + \sigma SdW)x - \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt - \sigma S\frac{\partial C}{\partial S}dW$

Since, $x = \frac{\partial C}{\partial S}$

Therefore,

$$(r + \delta)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r + \delta)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$\Rightarrow (r + \delta)C = (r + \delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} \qquad (A1)$$

The above is the analogy based PDE.

Appendix B

The analogy based PDE derived in Appendix A can be solved by converting to heat equation and exploiting its solution.

Start by making the following transformation:

$$\tau = \frac{\sigma^2}{2}(T - t)$$

$$x = ln\frac{S}{K} => S = Ke^x$$

$$C(S,t) = K \cdot\ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} - \ K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (A1) and writing $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$, we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1)\frac{\partial c}{\partial x} - \tilde{r}c \qquad (B1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S-K,0\}\ becomes\ c(x,0) = max\{e^x - 1,0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r}-1) - \tilde{r} - \beta)u + \big(2\alpha + (\tilde{r}-1)\big)\frac{\partial u}{\partial x} \qquad (B2)$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4}$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (B3)$$

With the initial condition:

$$u(x_0,0) = max\{\left(e^{(1-\alpha)x_0} - e^{-\alpha x_0}\right), 0\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}}u(x_0,0)dx_0$$

Change variables: $= \frac{x_0-x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0 \ iff \ x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})}dz - \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})}dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\left(x+z\sqrt{2\tau}\right)-\frac{z^2}{2}=-\frac{1}{2}\left(z-\frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2+\frac{\tilde{r}+1}{2}x+\tau\frac{(\tilde{r}+1)^2}{4}$$

$$=:-\frac{1}{2}y^2+c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1=\frac{e^c}{\sqrt{2\pi}}\int_{-x/\sqrt{2\pi}-\sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}}dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{y^2}{2}}dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{{}^{\tau}/_{2}}\,(\tilde{r}+1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{{}^{\tau}/_{2}}\,(\tilde{r}-1)$ and $f = \frac{\tilde{r}-1}{2}x+\tau\frac{(\tilde{r}-1)^2}{4}$

The analogy based European call pricing formula is obtained by recovering original variables:

$$Call = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)$$

Where$\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ $\boldsymbol{and\ d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$

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