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Dynamic awareness and zero probability beliefs

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Abstract

In this note, it is shown that in a Bayesian model with unawareness of impossible (probability zero), or vanishingly improbable, events, awareness can only change after such an improbable event has actually been observed.

I. Dynamic awareness and zero probability beliefs

Decisions must be often be made in circumstances where decisionmakers are unaware of all the relevant possibilities. The problem of unawareness has been the subject of a rapidly growing literature A variety of approaches have been proposed including those of Grant and Quiggin (2013), Halpern and Rego (2014), Heifetz, Meier and Schipper (2006) and Karni and Viero (2013). Schipper (2016) provides a bibliograph. The problem of unawareness has assumed increasing importance in the light of the global financial crisis, which was unforeseen by most policymakers and market participants. The popular treatment and well-timed treatment of the topic by Taleb (2007) refers to such events as 'black swans'. The global financial crisis also raises the need to model the dynamics of changing awareness, and in particular the process by which decisionmakers become aware of previously unconsidered possibilities.

There is no general agreement on the appropriate way to represent awareness and unawareness even in a static context. Heifetz, Meier and Schipper (2006) present a model in which the state space considered by a decisionmaker is the codomain of a surjection with the full state space, available to an unboundedly rational decisionmaker, as the domain. This representation of unawareness is referred to by Grant and Quiggin (2013) as 'coarsening'. Grant and Quiggin propose an alternative approach, referred to as 'reduction' in which the state space considered by decision maker is a subset of the full space.

Only limited attention has been paid to the dynamics of awareness. The most notable contribution is the 'reverse Bayesianism' of Karni and Viero (2013). The central idea is that, when decisionmakers become aware of a new possible state of nature, and assign it a positive probability, the probabilities for all elements of the pre-existing state space are reduced proportionally.

An important question in this context is that of whether reduction unawareness may be managed within a standard Bayesian setting by treating events of which decisionmakers are unaware as having zero, or vanishingly small, prior probability, then being updated in the light of new information.. This question has been addressed, in passing, by authors including Grant and Quiggin (2013), Li (2008), Modica (2008) and Meier and Schipper (2013). All the authors cited have drawn distinctions between unawareness and zero probability beliefs. However, none has addressed the problem systematically.

In this note, it is shown that in a Bayesian model with unawareness of impossible (probability zero), or vanishingly improbable, events, awareness can only change after such an improbable event has actually been observed.

II. Notation

We consider a finite set of states S, with typical element s, and the associated algebra of events $\mathcal{S} = 2^{\Omega}$, with typical element $E \subseteq S$. The use of a finite state space avoids complications associated with sets of measure zero. Moreover, in the context of bounded rationality, it makes sense to impose the requirement that finite agents can only be aware of finitely many possibilities.

To represent awareness we will need to consider algebras $\mathcal{F}$, $\mathcal{G} \subset \mathcal{S}$. Note that we do not require that $S \in \mathcal{F}$. More precisely, for any $\mathcal{F} \subseteq \mathcal{S}$, we will call the maximal element

$$F = \bigcup_{E \in \mathcal{F}} E$$

the *scope* of $\mathcal{F}$. F contains all states which are elements of some event E in $\mathcal{F}$. Note, however that we do not require $s \in F \Rightarrow \{s\} \in \mathcal{F}$.

We define

Definition 1. Let $\mathcal{F}$, $\mathcal{G} \subset \mathcal{S}$ be algebras of subsets of S, with $\mathcal{G} \subset \mathcal{F}$.We say that

(i) $\mathcal{F}$ is a refinement of $\mathcal{G}$ if $G = F$, that is if the two algebras have the same scope

(ii) $\mathcal{F}$ is an expansion of $\mathcal{G}$ if $\mathcal{F} \cap G = \mathcal{G}$, that is if the two algebras coincide on the scope of

$\mathcal{G}$

Conversely, if (i) holds we will say that $\mathcal{G}$ is a coarsening of $\mathcal{F}$ and if (ii) holds, we will say that $\mathcal{G}$ is a reduction of $\mathcal{F}$. These terms correspond to the alternative concepts of unawareness distinguished by Grant and Quiggin (2012). With this in mind, we will define $\mathcal{A} \subset \mathcal{S}$ to be a algebra consisting of events in $\mathcal{S}$ of which an agent is aware, and denote the scope of $\mathcal{A}$ by

$$A = \bigcup_{E \in \mathcal{A}} E$$

Conversely, we will say that the agent is unaware of events in $\mathcal{U} = \mathcal{S} - \mathcal{A}$, noting that $\mathcal{U}$ is not, in general, an algebra.

Define an information partition $\mathcal{I}$ on $\mathcal{S}$ as a set of mutually exclusive, exhaustive non-trivial events $\{I : I \in \mathcal{I}\} \subset \mathcal{S} - \varnothing$, referred to as signals. To simplify the statement of results, we will assume that any information partition has at least two (non-trivial) elements.

We say that the decisionmaker is aware of the information partition $\mathcal{I}$ whenever $\mathcal{I} \cap A \subset \mathcal{A} - \varnothing$. That is, for each signal $I \in \mathcal{I}$, $\mathcal{I} \cap A$ is a non-empty event of which the agent is aware. Observe that, in this case, $\mathcal{I} \cap A$ represents an information partition for $\mathcal{A}$. Given an information partition $\mathcal{I}$ we define a general updating operator $\|$. For any $I \in \mathcal{I}$ let $\mathcal{A}\|I$ denote the algebra of events of which the agent is aware conditional on the signal I and define $A\|I$ and $U\|I$ correspondingly. We will say that $I \in \mathcal{I}$ generates an expansion (contraction) of awareness if $\mathcal{A} \subset \mathcal{A}\|I$ ($\mathcal{A}\|I \subset \mathcal{A}$).

A. Probabilistic awareness and Bayesian updating

For a large class of models, beliefs are interpreted in terms of probabilities. We will denote by $p : \mathcal{S} \to [0, 1]$ a 'prior' probability distribution on $\mathcal{S}$ in the absence of any information and let $p_{\mathcal{A}} : \mathcal{A} \to [0, 1]$ be the restriction of p to $\mathcal{A}$. Then, for any information partition $\mathcal{I}$ and $I \in \mathcal{I}$ we let $p_{\mathcal{A}\|I} : \mathcal{A}\|\mathcal{I} \to [0, 1]$ be the probability distribution on $\mathcal{A}\|I$ following the observation of I. We refer to $\|$ as an updating rule.

The concept of awareness precludes the assignment of positive probability to an event of which an agent is unaware. Conversely, awareness of a contingency relevant to a decision requires consideration of some state of the world consistent with that event, which in turn requires that the event must have non-zero probability.

Definition 2. An agent displays probabilistic awareness if $p(S - A) = 0$. An agent displays strict probabilistic awareness if $p(E) > 0$, $\forall E \in \mathcal{A}$

The standard Bayesian model is consistent with strict probabilistic awareness. We now consider the updating of awareness in the Bayesian context. The defining feature of the Bayesian model is Bayes' rule $\|^B$

$$p(E|I) = \frac{p(I \cap E)}{p(I)} \tag{1}$$

Applying this to dynamic awareness, we require that, for all E in $\mathcal{A}\|I \cap \mathcal{A}$, $P_{\mathcal{A}\|I}(E)$ satisfies equation (1). The Bayesian updating rule is $\|^B$ is well defined for I if and only if the decisionmaker is aware of I. Further, $p(E) = 0 \Rightarrow p_{\mathcal{A}\|^B I}(E|I) = 0$. In particular, since $p(S - A) = 0$, we have $p_{\mathcal{A}\|^B I}(S - A) = 0$.

Strict probabilistic awareness requires that for all

$$\mathcal{A}\|^B I = \left\{E \in \mathcal{A} : p_{\mathcal{A}\|I}(E|I) > 0\right\}$$

This yields:

Proposition 1. *For any information partition $\mathcal{I}$ of which the agent is aware, any $I \in \mathcal{I}$ generates a contraction of awareness under the Bayesian updating rule* $\|^B$

This proposition reflects the nature of information and updating in the Bayesian framework. Given a state space S and a partition $\mathcal{I}$, the observation of any $I \in \mathcal{I}$ amounts to the assignment of probability zero to all states in $S - I$. Taking this process to its limit, the realization of some state s implies that assignment of probability 1 to s and probability zero to $S - \{s\}$. The process of Bayesian updating consists entirely of successive contractions of the set of states of nature under consideration.

The converse of Proposition 1 is more interesting

Proposition 2. *Under the Bayesian rule* $\|^B$ *an expansion of awareness can arise only as a result of the observation of a signal with prior probability zero.*

That is, Bayesian updating allows the assignment of positive probability to an event previously considered impossible (assigned probability zero) only if such an impossible event has already occurred.

B. Incredible claims require incredible evidence

We now consider how the results derived above change if we consider events that are *a priori* 'incredible', rather than requiring a subjective probability of zero. To formalize this idea, and allow the use of asymptotic notation, we will introduce a parameter υ representing the subjective level of unawareness, and consider an event E that becomes incredible as $\upsilon \to 0$. More precisely, we allow $p(E)$ to depend on υ and require that $p(E;\upsilon)$ is $o(\upsilon)$ as $\upsilon \to 0$. That is,

$$\lim_{\upsilon \to 0} \frac{p(E;\upsilon)}{\upsilon} = 0 \tag{2}$$

We will use the notation $O(\upsilon)$ to indicate a variable x (or non-zero constant) such that

$$\lim_{\upsilon \to 0} \frac{x}{\upsilon} > 0 \tag{3}$$

Now consider the observation of some information I with prior probability $p(I)$. Bayes' rule gives

$$p(E|I) = \frac{p(I|E)p(E)}{p(I)} < \frac{p\left(E\right)}{p\left(I\right)} \tag{4}$$

and, conversely

$$p(E^c|I) = \frac{p(I|E^c)p(E^c)}{p(I)} > \frac{p(I) - p\left(E\right)}{p\left(I\right)} = 1 - p(E)p(I)$$

??.

The immediate implication is that for any $p(I)$, that is $O\left(\upsilon\right)$, $p(E|I)$ is $o\left(\upsilon\right)$.

That is, in a Bayesian model, events that are incredible *a priori* remain incredible *a posteriori* in the face of any evidence that is itself credible. To put this less formally 'incredible claims require incredible evidence'. This discussion leads us naturally to the converse proposition

Proposition 3. *Given strict probabilistic awareness, a signal $I \in \mathcal{I}$ generates an expansion of awareness under an updating rule $\|$ if and only if there exists some event E such that* $p\left(E\right) = 0$, $p_{\mathcal{A}\|I}\left(E|I\right) > 0$.

III. Concluding comments

In terms of static decision theory, being unaware of an event is the same as assigning zero probability to that event. Thus, consideration of unawareness does not require any adjustment to the standard Bayesian approach. However, as has been shown in this note, the Bayesian interpretation of unawareness in terms of zero (or incredibly small) probabilities is inconsistent with a dynamic model in which awareness changes over time. In essence, a Bayesian can become aware of a zero-probability event only if such an event has already occurred.

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