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Invariant risk attitudes

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Abstract

Concepts of constant absolute risk aversion and constant relative risk aversion have proved useful in the analysis of choice under uncertainty, but are quite restrictive, particularly when they are imposed jointly. A generalization of constant risk aversion, referred to as invariant risk aversion is developed. Invariant risk aversion is closely related to the possibility of representing preferences over state-contingent income vectors in terms of two parameters, the mean and a linearly homogeneous, translation-invariant index of riskiness. The best-known index with such properties is the standard deviation. The properties of the capital asset pricing model, usually expressed in terms of the mean and standard deviation, may be extended to the case of general invariant preferences.

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1. Introduction

A standard problem in consumer theory is the need to distinguish between income and substitution effects. Because substitution effects are usually of primary interest, much attention has been paid to various homotheticity and quasi-homotheticity concepts under which income effects can be either disregarded or distinguished in a fairly simple fashion.

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In the analysis of choice under uncertainty, a similar role has been played by the concepts of constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA). When CARA holds, changes in wealth have no effect on preferences over risky prospects. When CRRA holds, demands for risky assets are linear in wealth.

In the expected-utility (EU) literature, the notions of CARA and CRRA have been closely linked to the coefficients of absolute and relative risk aversion derived from the utility function. Perhaps because of this linkage, a similar analysis has been slow to develop for more general models of choice under uncertainty. A more promising starting point is the geometrical analysis of Brennan and Kraus [3] who show that CARA and CRRA for EU preferences both correspond to linear expansion paths. Chambers and Färe [4] and Quiggin and Chambers [18] show that, when preferences are modelled in terms of state-contingent outcome vectors, CARA and CRRA correspond naturally to standard concepts of translation homotheticity and radial homotheticity [8], respectively. Thus the results of Brennan and Kraus are valid without the assumption of EU preferences.

It is natural to consider the possibility that preferences may display both CARA and CRRA. One approach, developed by Meyer [14] and Sinn [24] is to restrict attention to choice sets in which any element can be derived from any other by a combination of a translation and a scale shift. However, this approach is only applicable to a limited range of problems.

In EU theory, it is obvious from the nature of the coefficients of absolute and relative risk aversion [17] that, for strictly concave preferences, constant relative risk aversion implies decreasing absolute risk aversion. Hence, EU preferences display both CARA and CRRA only in the trivial case of risk neutrality.

Safra and Segal [22] and Quiggin and Chambers [18] examine this question for more general preferences. Safra and Segal use the term *constant risk aversion* to refer to preferences that display both CARA and CRRA, and characterize constant risk aversion for a range of generalized EU models. For example, Safra and Segal show that the combination of constant risk aversion, preference for diversification and ‘zero-independence’ (a generalization of the homogeneity assumption of Rubinstein et al. [21] and Grant and Kajii [9]) is equivalent to the requirement that preferences must be representable by the S-Gini functional form of Donaldson and Weymark [7], which is equivalent to the dual model of Yaari [26]. Quiggin and Chambers give a functional characterization of constant risk aversion and extend the results of Pratt [17] by showing that CRRA must be associated with non-increasing absolute risk aversion, implying that constant risk aversion represents a polar case.

Although the case of constant risk aversion is of considerable interest, preferences satisfying constant risk aversion display a number of characteristics that may prove unattractive in some applications. As shown by Safra and Segal [22], strictly risk-averse preferences satisfying constant risk aversion cannot be smooth. Another problem arises in relation to asset demands. As Yaari [26] observes, preferences in the dual model display ‘plunging’ behavior. That is, in any choice problem in which a given amount of wealth must be allocated between a safe asset and one or more risky assets, with no borrowing or short-selling, any unique solution will involve either all wealth being allocated to the safe asset or none. Chambers and Quiggin [6] show that this plunging behavior is characteristic of the entire class of quasi-concave, constant risk averse preferences.

The main object of this paper is to consider a concept less stringent than constant risk aversion that retains many of the features that make constant risk aversion a useful analytical tool. Preferences are described as translation (radially) invariant if the ranking of two state-contingent income vectors with equal mean is not affected by a change in base wealth (change in scale).

We begin by introducing a range of characterizations of preferences, and use these to define CARA, CRRA and a closely related concept referred to as linearity of preferences on equal-mean sets, or, more briefly, linearity. It is shown that if preferences satisfy any two of CARA, CRRA and linearity, they satisfy the third, and therefore display constant risk aversion.

In the main part of the paper, we introduce three notions of invariance, generalizing, respectively, CARA, CRRA and linearity. Analogously with the results for constant risk aversion, it is shown that if preferences satisfy any two of the three invariance conditions, they satisfy the third. Preferences satisfying all three invariance conditions are referred to as *invariant*. We first show that the only EU preferences displaying invariance are quadratic preferences, that is, that EU preferences are invariant only where they coincide with mean-standard deviation preferences. We then consider general preference structures and show that preferences are invariant if and only if they can be expressed in terms of the mean and a *risk index*. The risk index generalizes the standard deviation while retaining many of its most tractable (and hence desirable) properties including positive linear homogeneity, subadditivity, translation invariance and positivity. It attains its lower bound at the traditionally riskless outcome. Invariant preferences are also shown to generalize directly the class of CRA preferences. Applications include an extension of the capital asset pricing model (CAPM) to the case of general invariant preferences.

2. Notation and assumptions

The analytical tools used in this paper are those developed in detail by Quiggin and Chambers [18] and Chambers and Quiggin [6], drawing on standard methods of convex analysis [2,19]. The analysis in this paper is presented in primal terms. However, as is shown in [6], a dual approach extending that of Peleg and Yaari [16] (see also [15]) is equally applicable and, in some cases, more powerful.

We concern ourselves with preferences over random state-contingent income or consumption, represented as measurable mappings from a probability measure space $\Omega = (S, \Sigma, P)$ to an outcome space $Y \subseteq \Re$. We emphasize the vector space structure of the space of mappings Y^S, and, in particular, denote by $\mathbf{1}$ the constant mapping such that $\mathbf{1}(s) = 1,\ \forall s \in S$. Preferences over Y^S are given by a total ordering with an associated indifference ordering I that may be represented by a continuous, strictly increasing, quasi-concave, certainty equivalent function $e: Y^S \to \Re$. A certainty equivalent is a function representing I and satisfying the *agreement property* [1], that, for any $c, e(c\mathbf{1}) = c$.

Quasi-concavity ensures that the least-as-good sets of the preference mapping

$$V(c) = \{\mathbf{y} : e(\mathbf{y}) \geqslant c\}$$

are convex. We will also define the indifference set

$$I(c) = \{\mathbf{y} : e(\mathbf{y}) = c\}.$$

We denote the mean by

$$E(\mathbf{y}) = \int_S y(s)\, dP(s)$$

and observe that the mean has the properties, respectively, of translatability and linear homogeneity

$$E(\mathbf{y} + \delta\mathbf{1}) = E(\mathbf{y}) + \delta,$$

$$E(t\mathbf{y}) = tE(\mathbf{y}).$$

We further assume that preferences are *risk-averse* in the sense that

$$E(\mathbf{y}) - e(\mathbf{y}) \geqslant 0.$$

Preferences are strictly risk-averse if $e(\mathbf{y}) < E(\mathbf{y})$, for $\mathbf{y} \neq e(\mathbf{y})\mathbf{1}$.[1]

For any μ, we define the equal-mean set

$$M(\mu) = \{\mathbf{y} : E(\mathbf{y}) = \mu\}.$$

3. Constant risk aversion

The definitions above yield simple characterizations of constant absolute and relative risk aversion [18]. Preferences display *constant absolute risk aversion* (CARA) if

$$e(\mathbf{y} + \delta\mathbf{1}) = e(\mathbf{y}) + \delta, \quad \mathbf{y} \in Y^S, \ \delta \in \Re$$

and *constant relative risk aversion* (CRRA) if,

$$e(t\mathbf{y}) = te(\mathbf{y}), \quad \mathbf{y} \in Y^S_{++}, \ t \in \Re_{++},$$

where Y^S_{++} is the strictly positive orthant of Y^S. Following Safra and Segal [22], we say that preferences display *constant risk aversion* if they display both CARA and CRRA. Chambers and Quiggin [6] completely characterize, in both dual and primal

[1] Yaari [25] was the first to focus on this definition independently of its characterization in EU theory in terms of concavity of the utility function. For monotonic EU preferences, risk aversion in this sense is equivalent to aversion to increases in risk in the sense of Rothschild and Stiglitz [20], or equivalently to preservation of second-order stochastic dominance in the sense of Hadar and Russell [10] and Hanoch and Levy [11]. For more general preferences, this equivalence does not hold.

form, preferences that exhibit constant risk aversion. Note that CRRA, and therefore also constant risk aversion, is consistent with strict monotonicity on Y^S only if $Y \subseteq \Re_{++}$.

Finally, we define *linearity of preferences on equal-mean sets* by the requirement

$$e(\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}) = \lambda e(\mathbf{y}) + (1-\lambda)E(\mathbf{y}), \quad \lambda \in \Re_+.$$

The crucial observation which yields the relationship between CARA, CRRA and linearity on equal-mean sets is that, for $\lambda > 1$, any multiplicative spread around the mean from $\mathbf{y}$ to $\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}$ can be formed as the composition of a radial expansion (with expansion factor λ) and a (negative) translation of $(1-\lambda)E(\mathbf{y})\mathbf{1}$. Similarly, for $\lambda \in [0,1]$, any multiplicative contraction from $\mathbf{y}$ to $\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}$ can be formed as the composition of a radial contraction (with contraction factor λ) and a positive translation of $(1-\lambda)E(\mathbf{y})\mathbf{1}$.

Result 1. *For a certainty equivalent e, defined on Y_{++}^S, any two of*

(i) *CARA*,
(ii) *CRRA*, *and*
(iii) *linearity on equal-mean sets imply the third.*

Proof. All proofs are in the appendix. □

A certainty equivalent e displays constant risk aversion if and only if, for any c,

$$e(\lambda\mathbf{y} + (1-\lambda)c\mathbf{1}) = \lambda e(\mathbf{y}) + (1-\lambda)c, \quad \lambda, c \in \Re_+.$$

Hence, under constant risk aversion, e is homothetic in the usual radial sense, and also translation homothetic in the direction of the certainty ray, $\mathbf{1}$.

Both CARA and CRRA are convenient simplifications, allowing analysis to abstract from wealth and scale effects, respectively. However, as the analyses of Safra and Segal [22] and Chambers and Quiggin [6] show, the combination of CARA and CRRA can only be realized under restrictive conditions. In particular, quasi-concave preferences are consistent with constant risk aversion if and only if they correspond to the support function for a closed subset of the probability simplex, or perhaps more intuitively, if and only if they belong to the maxmin expected-value (MMEV) class [22]:

$$e(\mathbf{y}) = \inf_{\boldsymbol{\pi}} \{\boldsymbol{\pi}\mathbf{y} : \boldsymbol{\pi} \in \hat{\mathscr{P}}\} \tag{1}$$

for some closed convex $\hat{\mathscr{P}} \subseteq P$, where $\mathscr{P} \subset \Re_+^S$ denotes the probability simplex.

4. Invariance

CARA implies that judgements of comparative riskiness are unaffected by translations, that is, by changes in base wealth. Similar analogies apply with respect to relative risk aversion and risk aversion on equal-mean sets $M(\mu)$.

We say that preferences are: *translation-invariant for equal-mean sets* if, for all $\delta \in \Re, \mu \in \Re, \mathbf{y}, \mathbf{y}' \in M(\mu)$,

$$e(\mathbf{y}) \geqslant e(\mathbf{y}') \Rightarrow e(\mathbf{y} + \delta\mathbf{1}) \geqslant e(\mathbf{y}' + \delta\mathbf{1});$$

radially invariant for equal-mean sets if, for all $t \in \Re_+, \mu \in \Re, \ \mathbf{y}, \mathbf{y}' \in M(\mu)$,

$$e(\mathbf{y}) \geqslant e(\mathbf{y}') \Rightarrow e(t\mathbf{y}) \geqslant e(t\mathbf{y}');$$

and *multiplicative spread invariant for equal-mean sets* if, for all $\lambda \in \Re_+$, $\mu \in \Re, \ \mathbf{y}, \mathbf{y}' \in M(\mu)$,

$$e(\mathbf{y}) \geqslant e(\mathbf{y}') \Rightarrow e(\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1}) \geqslant e(\lambda\mathbf{y}' + (1-\lambda)\mu\mathbf{1}).$$

In what follows, we will refer to these properties simply as translation invariance, radial invariance and multiplicative-spread invariance, omitting the explicit reference to equal-mean sets.

Note that although preferences displaying CARA (CRRA, linearity on equal-mean sets) are translation-invariant (radially invariant, multiplicative spread invariant), the converse is not true. The example of mean-standard deviation preferences provides an illustration. Consider any preference function of the form

$$e(\mathbf{y}) = \phi(E(\mathbf{y}), \sigma(\mathbf{y})), \tag{2}$$

where we assume that

$$E(\mathbf{y}) = \int_S y(s)\, dP(s),$$

$$\sigma(\mathbf{y}) = \sqrt{\int_S (y(s) - E(\mathbf{y}))^2\, dP(s)}$$

and that ϕ is increasing in its first argument and decreasing in its second with $\phi(c, 0) = c$. For such preferences and $\mathbf{y}, \mathbf{y}' \in M(\mu)$, $e(\mathbf{y}) \leqslant \mathbf{e}(\mathbf{y}')$ if and only if $\sigma(\mathbf{y}) \leqslant \sigma(\mathbf{y}')$. However, as Quiggin and Chambers [18] show, these preferences are consistent with constant risk aversion (CRA) only if ϕ is linear.

Any positive translation can be formed as the composition of a radial expansion and a multiplicative contraction, and any radial expansion (or contraction) can be formed as the composition of a translation and a multiplicative spread (contraction). Hence, we have the following analog of Result 1.

Result 2. *Any two of* (i) *translation invariance*, (ii) *radial invariance*, *and* (iii) *multiplicative spread invariance imply the third.*

In view of this result, preferences that are both translation invariant and radially invariant will be referred to simply as *invariant*.

EU preferences, with a von Neumann–Morgenstern utility function u, are translation-invariant (radially invariant) if u displays CARA (CRRA). Also since $e(\mathbf{y}) \leqslant e(\mathbf{y}') \Leftrightarrow \sigma(\mathbf{y}) \leqslant \sigma(\mathbf{y}')$ whenever u is quadratic, EU preferences are invariant in this case. In fact, it may be shown that the quadratic is the only differentiable expected utility function with this property.

Result 3. *Differentiable EU preferences, with a von Neumann–Morgenstern utility function u, are invariant if and only if u is quadratic.*

4.1. The risk index

We have already noted that the class of mean-standard deviation models provide an example of a class of invariant preferences. A convenient feature of mean-standard deviation preferences is the fact that they can be represented by just two summary statistics for the state-contingent consumption distribution, one representing expected return and the other representing dispersion about the mean. We now show that this general property exactly characterizes the class of invariant preferences. That is, we demonstrate that invariant preferences can always be written in the general form

$$e(\mathbf{y}) = \phi(E(\mathbf{y}), \rho(\mathbf{y})),$$

where $\rho(\mathbf{y})$ has many of the same convenient properties as the standard deviation, namely, positive linear homogeneity, convexity and translation invariance.

Define

$$K(\mathbf{y}) = M(E(\mathbf{y})) \cap V(e(\mathbf{y}))$$

and note that for $\mathbf{y}, \mathbf{y}' \in M(\mu)$, $e(\mathbf{y}) \geqslant e(\mathbf{y}')$ if and only if $\mathbf{y} \in K(\mathbf{y}')$.

For the special case of mean-standard deviation preferences, the set $K(\mathbf{y})$ is defined by the equations $\hat{\mathbf{y}} \epsilon K(y)$ if

$$\int_S y(s)\, dP(s) = E(\hat{\mathbf{y}})$$

and

$$\sqrt{\int_S (\hat{y}(s) - E(\hat{\mathbf{y}}))^2\, dP(s)} \leqslant \sigma(\mathbf{y}).$$

In this case, for any possible $\mathbf{y}' \in M(E(\mathbf{y}))$, the set $K(\mathbf{y}')$ is simply a radial expansion or contraction of $K(\mathbf{y})$ around the point $E(\mathbf{y})\mathbf{1}$. Also, for any μ, μ', the families $\{K(\mathbf{y}) : \mathbf{y} \in M(\mu)\}$ and $\{K(\mathbf{y}) : \mathbf{y} \in M(\mu')\}$ are related by a simple translation. These properties hold more generally for translation-invariant preferences.

More precisely, for translation-invariant preferences

$$K(\mathbf{y} + \delta\mathbf{1}) = K(\mathbf{y}) + \delta\mathbf{1}, \quad \delta \in \Re.$$

For radially invariant preferences, and for $\mathbf{y} \in \Re^S_+$,

$$K(t\mathbf{y}) = tK(\mathbf{y}), \quad t > 0.$$

For multiplicative-spread invariant preferences,

$$K(\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}) = \lambda K(\mathbf{y}) + (1-\lambda)E(\mathbf{y})\mathbf{1}.$$

If preferences are strictly quasi-concave, for any $\mathbf{y}, \mathbf{y}'$ neither of which lies on the ray $e\mathbf{1}$, there must exist a unique t such that

$$E(\mathbf{y}')\mathbf{1} + \frac{(\mathbf{y} - E(\mathbf{y})\mathbf{1})}{t} \in K(\mathbf{y}').$$

Define, for $\mathbf{y}' \neq e\mathbf{1}$, the function $r(\bullet; \mathbf{y}') : Y^S \to \Re$

$$r(\mathbf{y}; \mathbf{y}') = \inf\{t > 0 : \mathbf{y} - E(\mathbf{y})\mathbf{1} \in t(K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1})\}$$

if there is $\mathbf{y} - E(\mathbf{y})\mathbf{1} \in t[K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}]$, and ∞ otherwise.

We observe:[2]

Lemma 1. *For quasi-concave, risk-averse preferences e, and* $\mathbf{y}' \neq e\mathbf{1}$, $r(\mathbf{y}; \mathbf{y}')$ *is positively linearly homogeneous, subadditive and lower semicontinuous in* $\mathbf{y}$. *If preferences are strictly risk-averse,* $r(\mathbf{y}; \mathbf{y}')$ *is a norm on* $M(0)$, *the space of zero-mean distributions.*

We next observe on the basis of the fundamental polarity between gauge and support functionals that:

Result 4. *For quasi-concave, risk-averse preferences e, and* $\mathbf{y}' \neq e\mathbf{1}$, $r(\mathbf{y}; \mathbf{y}')$ *can always be written in the form*

$$r(\mathbf{y}, \mathbf{y}') = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - \mathrm{E}(\mathbf{y})1) : \mathbf{p} \in K^*\},$$

where K^* *is a closed convex set that contains the origin.*

Result 4 offers a workable alternative for specifying $r(\mathbf{y}, \mathbf{y}')$ without direct reliance on a particular choice of $\mathbf{y}'$ and $K(\mathbf{y}')$. Simply specify a closed, convex set K^* containing the origin. Then $r(\mathbf{y}, \mathbf{y}')$ can be deduced directly as the support function for K^*. The associated $K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}$ can then be deduced by using the resulting $r(\mathbf{y}, \mathbf{y}')$ as a gauge function, as illustrated in the proof of the lemma. Because such a set will always exist, the choice of $\mathbf{y}'$ is irrelevant apart from simple issues of scaling. Hence, r may be treated as though it were independent of $\mathbf{y}'$.

The importance of the fact that r may be treated as though it were independent of the choice of $\mathbf{y}'$ may be appreciated by considering the case of mean–variance preferences. In general, any monotonic transformation of the variance can be used in the representation of such preferences. However, the only functions of $\mathbf{y}$ monotonically increasing in the variance of $\mathbf{y}$ that are both linearly homogeneous and translation-invariant are scalar multiples of the standard deviation. Thus, the standard deviation, and not the variance, provides the canonical representation of riskiness for this class of preferences. More generally, since the function r is independent of the choice of $\mathbf{y}'$ up to a scalar multiple, we may choose an appropriate normalization and define a *risk index* $\rho(\mathbf{y})$ unique up to a scalar multiple.

[2] We thank an anonymous referee for suggesting this point.

We are now in a position to prove the main result.

Result 5. *Quasi-concave, risk-averse preferences are invariant if and only if there exist functions* $\rho : Y^S \to \Re_+$ *and* $\phi : \Re \times \Re_+ \to \Re$, *such that*

(i) $$\rho(\mathbf{y}) = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - \mathrm{E}(\mathbf{y})\mathbf{1}); \mathbf{p} \in K^*\}$$

with K^* *a closed convex set, and*

(ii) $$e(\mathbf{y}) = \phi(E(\mathbf{y}), \rho(\mathbf{y})),$$

with ϕ *increasing in its first argument and decreasing in its second and* $\phi(c, 0) = c$.

Several observations on Result 5 are pertinent. First, ρ, by the properties of support functionals, is positively linearly homogeneous and subadditive (sublinear), and lower semi-continuous. Second, it is obviously translation invariant so that

$$\rho(\mathbf{y} + \delta\mathbf{1}) = \rho(\mathbf{y}), \quad \delta \in \Re.$$

ρ assumes its minimal value of 0 at points along the ray $c\mathbf{1}$, that is, the traditionally riskless portfolios.

Example 1. Let $K^* = \{\mathbf{p} : ||\mathbf{p}|| \leqslant 1\}$, where $||\mathbf{p}||$ is the Euclidean norm. The Euclidean norm is sublinear and lower semicontinuous. Thus, $||\mathbf{y}||$ is a support function for the set

$$\{\mathbf{p} : \mathbf{py} \leqslant ||\mathbf{y}|| \text{ for all } \mathbf{y}\},$$

so that, by the Cauchy–Schwarz inequality [19],

$$||\mathbf{y}|| = \sup_{\mathbf{p}} \{\mathbf{py} : ||\mathbf{p}|| \leqslant 1\}$$
$$= \sup_{\mathbf{p} \in K^*} \{\mathbf{py}\}.$$

Hence, the risk index associated with K^* is

$$||\mathbf{y} - E(\mathbf{y})\mathbf{1}|| = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : ||\mathbf{p}|| \leqslant 1\},$$

which clarifies the invariant nature of mean-standard deviation preferences.

Example 2. Let $K^* = \{\mathbf{p} : p(s) \in [-1, 1],\ s \in \Omega\}$. The associated risk index is

$$\rho(\mathbf{y}) = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : p(s) \in [-1, 1], s \in \Omega\}$$
$$= |\mathbf{y} - E(\mathbf{y})|\mathbf{1},$$

where $|\mathbf{y} - E(\mathbf{y})| \in Y^S$ denotes the vector of absolute deviations from the mean. Hence, the class of mean absolute deviation (MAD) preferences are invariant.

4.1.1. Constant risk aversion and invariance

By Result 5, ϕ must be decreasing in ρ. Thus,

Corollary 2. *Quasi-concave, risk-averse preferences are invariant if and only if there exists a function* $\phi : \Re \times \Re_+ \rightarrow \Re$, *such that*

$$e(\mathbf{y}) = \inf_{\mathbf{p}}\{\phi(E(\mathbf{y}), \mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1})) : \mathbf{p} \in K^*\},$$

where K^* *is a closed, convex set containing the origin, and* ϕ *is nondecreasing in its first argument, nonincreasing in its second, and* $\phi(c, 0) = c$.

Invariant preferences can thus be linked intuitively to the class of CRA preferences. We have the following result that relates CRA preferences and invariant preferences, and generalizes an earlier result of Quiggin and Chambers [18] relating CRA and mean-standard deviation preferences.

Result 6. *Invariant references display constant risk aversion if and only if the certainty equivalent can be written in the form*

$$e(\mathbf{y}) = E(\mathbf{y}) - k\rho(\mathbf{y}).$$

Hence, for the class of CRA preferences the risk index is a simple multiple of the absolute risk premium. Well-known members of the CRA class include maxmin, linear mean-standard deviation preferences, and Yaari dual preferences, as well as the class of linear mean absolute deviation (MAD) preferences. Because there are an infinity of closed, convex sets containing the origin, there is also an infinite number of members of the invariant class.

The hypothesis of invariance requires that the function ρ, which may be regarded as an index of the riskiness of $\mathbf{y}$, should be translation invariant and positively linearly homogeneous. Constant risk aversion, on the other hand, requires that the absolute risk premium be translation invariant and positively linearly homogeneous. This extra flexibility allows for a wide range of possible wealth effects. It follows immediately from these observations that, given a class of invariant preferences, one can always use that risk index to induce an associated class of CRA preferences. Conversely, if one has a class of CRA preferences, one can always use the associated absolute risk premium to induce an associated class of invariant preferences.

4.1.2. Invariance and quasi-homotheticity

The class of preferences exhibiting hyperbolic absolute risk aversion (HARA), also known as linear risk tolerance, has received much attention in the literature on expected utility functionals. A number of widely used functional forms, including exponential, power and quadratic utility functions satisfy HARA. They have assumed particular importance because in the standard portfolio problem, optimal asset holdings for these preferences are linear in the individual's wealth. In other words, these preferences are characterized by linear expansion paths. For more general preferences, linearity of expansion paths is a defining characteristic of quasi-homotheticity. Chambers and Quiggin [6] have used this observation as the basis for a definition of linear risk tolerance for general choice functionals.

As will be shown below, invariant preferences, like quasi-homothetic preferences display a form of two-fund separation in portfolio problems with a safe asset. Lewbel and Perraudin [12] have shown that rank-two demand systems, which incorporate the demands arising from quasi-homothetic preferences, satisfy the conditions for two-fund separation in portfolio problems. Quasi-homothetic preferences display linear risk tolerance (the expansion curve is linear), whereas for invariant preferences the expansion curve may be any one-dimensional manifold in $\Re^2$. On the other hand, except in trivial cases where all expansion curves have common slope, quasi-homothetic preferences do not yield a tractable model of asset pricing, while, as shown below, invariant preferences do.

A prominent member of the class of preferences satisfying both invariance and linear risk tolerance is the CRA preference family. Another is the class of EU preferences with quadratic utility. These preferences are obviously quasi-homothetic, and, as demonstrated above, they also satisfy invariance. Thus, EU preferences with quadratic utility may be viewed as maximally tractable. This maximal tractability emerges from the simultaneous imposition of invariance, quasi-homotheticity, and state-wise separability on the choice functional. It is not surprising, therefore, that quadratic loss functions have historically played such an important role in economic analysis. On the other hand, considered as a utility function over wealth, the quadratic has a number of unattractive properties including increasing absolute risk aversion (assuming preferences are not risk-seeking). These unattractive properties are the consequence of the simultaneous imposition of so many restrictions on preferences. Chambers and Quiggin [5] have characterized in dual and primal terms the class of general choice functionals satisfying both linear risk tolerance and invariance. Their results enable one to maintain much of the maximal tractability of quadratic-EU preferences, while allowing one to avoid some of their more unattractive characteristics.

5. Asset demand

We now consider the implications of invariant preferences for the problem of determining asset demand in the presence of a safe asset. We will normalize by assuming that all assets have price 1. Let $\mathbf{y}^i \in \Re^S$, $i = 1, \dots, N$ be the return vector on asset i, let $\tilde{\mathbf{Y}} \in \Re^{S\times N}$ be the associated returns matrix, let $\alpha_i \geqslant 0$ be the holding of asset i, and let W be initial wealth. Denote by $\boldsymbol{\alpha}$ the vector $(\alpha_1, \dots, \alpha_N)$. Let asset 1 be the safe asset, and denote its return by r, so that $\mathbf{y}^1 = r\mathbf{1}$. Let $\mathbf{e}^1 = (1, 0, 0, \dots, 0)$ denote the vector corresponding to a portfolio consisting of one unit of the safe asset, and for any $\boldsymbol{\alpha}$, let

$$\boldsymbol{\alpha}_{-1} = \boldsymbol{\alpha} - \alpha_1 \mathbf{e}^1 = (0, \alpha_2 \dots, \alpha_N)$$

denote the risky component of the portfolio. The portfolio problem is to determine

$$\max_{\boldsymbol{\alpha}} \left\{ \phi(E(\mathbf{y}), \rho(\mathbf{y})) : \sum_{i=1}^{N} \alpha_i = W, \mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{y}^i \right\}. \tag{3}$$

In the presence of a riskless asset, invariant preferences display the property of two-fund separation familiar from the analysis of mean-standard deviation preferences in finance theory developed by Lintner [13] and Sharpe [23]. Just as in mean-standard deviation analysis, the returns matrix with the riskless asset deleted gives rise to a frontier representing the trade-off between expected returns $E(\mathbf{y})$ and risk $\rho(\mathbf{y})$. The tangent from the point (1,0) representing $\mathbf{e}^1$ then gives rise to a linear risk-return frontier.

To develop this point rewrite the basic problem after substitution as follows:

$$\begin{aligned}\max_{\boldsymbol{\alpha}}&\left\{\phi(E(\mathbf{y}),\rho(\mathbf{y})):\ \mathbf{y}=\left(W-\sum_{j=2}^{N}\alpha_j\right)r\mathbf{1}+\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right\}\\ &=\max_{\boldsymbol{\alpha}_{-1}}\left\{\phi\left(\left(W-\sum_{j=2}^{N}\alpha_j\right)r+E\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right),\rho\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)\right)\right\}.\end{aligned}$$

By the monotonicity properties of ϕ from Result 5, any solution to this problem is also a solution to the problem of maximizing expected returns for the optimal risk level ρ. This permits an informative decomposition of this problem. Consider the first-stage problem:

$$\begin{aligned}\max_{\boldsymbol{\alpha}_{-1}}&\left\{\left(W-\sum_{j=2}^{N}\alpha_j\right)r+E\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right):\rho\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)\leqslant\rho\right\}\\ &=Wr+\max_{\boldsymbol{\alpha}_{-1}}\left\{E\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)-r\sum_{j=2}^{N}\alpha_j:\rho\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)\leqslant\rho\right\}\\ &=Wr+V(\hat{\mathbf{Y}},\rho),\end{aligned}$$

so that $Wr+V(\hat{\mathbf{Y}},\rho)$ is the maximal expected value of the optimal portfolio given that the optimal risk level is ρ.

By the linear homogeneity of the risk index for $\rho>0$,

$$\begin{aligned}V(\hat{\mathbf{Y}},\rho)&=\max_{\boldsymbol{\alpha}_{-1}}\left\{E\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)-r\sum_{j=2}^{N}\alpha_j:\rho\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)\leqslant\rho\right\}\\ &=\max_{\boldsymbol{\alpha}_{-1}}\left\{E\left(\sum_{j=2}^{N}\alpha_j\mathbf{y}^j\right)-r\sum_{j=2}^{N}\alpha_j:\rho\left(\sum_{j=2}^{N}\frac{\alpha_j}{\rho}\mathbf{y}^j\right)\leqslant 1\right\}\\ &=\rho\max_{\frac{\boldsymbol{\alpha}_{-1}}{\rho}}\left\{E\left(\sum_{j=2}^{N}\frac{\alpha_j}{\rho}\mathbf{y}^j\right)-r\sum_{j=2}^{N}\frac{\alpha_j}{\rho}:\rho\left(\sum_{j=2}^{N}\frac{\alpha_j}{\rho}\mathbf{y}^j\right)\leqslant 1\right\}\\ &=\rho V(\hat{\mathbf{Y}},1).\end{aligned}$$

Thus, the optimal risky portfolio is linear in the level of the risk index, and $V(\hat{\mathbf{Y}},1)$ measures the marginal gain in expected return from assuming an additional unit of risk as measured by ρ. In equilibrium, therefore, $V(\hat{\mathbf{Y}},1)$ will equal the individual's marginal rate of substitution between the expected return and the risk index. Thus, the optimal risky portfolio can always be derived as the radial expansion, starting from $\mathbf{e}^1$, of the optimal risky portfolio for $\rho = 1$. We shall refer to solutions to $V(\hat{\mathbf{Y}}, \rho)$ as being on the *efficient frontier*.

We thus have:

Result 7. *For any given risk index* ρ, *and returns matrix* $\tilde{\mathbf{Y}}$ *including a riskless asset, the optimal portfolio for an individual with invariant preferences satisfies*

$$\boldsymbol{\alpha}_{-1} = \tilde{\rho}\tilde{\boldsymbol{\alpha}}_{-1},$$

$$\alpha_1 = W - \tilde{\rho}\sum_{j=2}^{N} \tilde{\alpha}_j,$$

where

$$\tilde{\boldsymbol{\alpha}}_{-1} \in \arg\max_{\boldsymbol{\alpha}_{-1}} \left\{ E\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right) - r\sum_{j=2}^{N} \alpha_j : \rho\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right) \leqslant 1 \right\}$$

and

$$\tilde{\rho} \in \arg\max_{\rho} \; \{\phi(Wr + \rho V(\hat{\mathbf{Y}}, 1), \rho)\}.$$

Comparative static analysis for the solution characterized in Result 7 is straightforward. The decision-maker's problem is to choose the efficient risk index to satisfy

$$\max_{\rho} \; \{\phi(Wr + \rho V(\hat{\mathbf{Y}}, 1), \rho)\}$$

and the usual single variable comparative statics apply.

It is also useful to consider the results derived above for the special case of constant risk aversion. Yaari [26] observes that preferences under his dual theory display 'plunging' behavior. That is, in any choice problem in which a given amount of wealth must be allocated between a safe asset and a risky asset, with no borrowing or short-selling, any unique solution will involve all wealth being allocated to one of the two assets. This is true more generally for preferences displaying constant risk aversion [6].

Under constant risk aversion, the analysis of portfolio problems involving a safe asset is straightforward. By Results 6 and 7, the portfolio problem can be written as

$$\operatorname*{Max}_{\rho} \; \{Wr + \rho V(\hat{\mathbf{Y}}, 1) - k\rho\}.$$

Because the individual's objective function is linear in the risk parameter, at the optimum he or she will set it equal either to zero or to its maximum feasible level. This, in turn, implies

Corollary 7.1. *Under constant risk aversion, and in the absence of short-selling and borrowing, if there exists a unique optimum, the optimal value of* α_1 *must be either* 0 *or* 1. *If there does not exist a unique optimum, all* $\alpha_1 \in [0, 1]$ *are weakly optimal.*

For an individual, Result 7 implies that asset choices can be represented by an expansion path in the two-dimensional space represented by holdings of the safe asset $\mathbf{e}^1$ and the efficient risky portfolio $\boldsymbol{\alpha}^*_{-1}$. In a market equilibrium, if all investors have invariant preferences of the form $\phi(E(\mathbf{y}), \rho(\mathbf{y}))$ for a common ρ, then $\boldsymbol{\alpha}^*_{-1}$ will be the market portfolio of risky assets, and all investors will hold mixtures of the riskless asset and the market portfolio. This observation leads to a straightforward method of pricing out financial assets.

The structure of the problem can be further exploited to develop this point further. For the optimal portfolio, α^*, the expected return on initial wealth is

$$r(\boldsymbol{\alpha}^*) = r + \frac{\rho\left(\sum_{j=2}^{N} \alpha_j^* \mathbf{y}^j\right)}{W} V(\hat{\mathbf{Y}}, 1).$$

By this construction, $V(\hat{\mathbf{Y}}, 1)$ is interpretable as the difference between the expected return on wealth and the riskless rate, normalized by the unit risk level, that is, the risk premium associated with the optimal portfolio.

Now consider any other portfolio $\boldsymbol{\alpha}$ on the efficient frontier which fully exhausts available wealth. Its return is

$$r(\boldsymbol{\alpha}) = r + \frac{\rho\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right)}{W} V(\hat{\mathbf{Y}}, 1),$$

whence

$$r(\boldsymbol{\alpha}) - r = \frac{\rho\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right)}{\rho\left(\sum_{j=2}^{N} \alpha_j^* \mathbf{y}^j\right)} [r(\boldsymbol{\alpha}^*) - r],$$

which naturally leads to asset pricing results entirely analogous to the standard CAPM model as derived from mean-standard deviation preferences.

Hence, much of the standard mean-standard deviation analysis can be extended to general invariant preferences, without requiring the restrictive and unappealing assumption that preferences are neutral with respect to skewness and higher moments of the distribution of returns. However, because the concept of invariance gives a crucial role to the certainty vector, the attractive separation properties derived above apply only in the presence of a safe asset.[3] In addition, because the Euclidean norm gives rise to an inner product, norms based on the standard deviation allow for a natural definition of notions such as covariance and correlation in terms of concepts of linear algebra. The corresponding analysis in the general case is nonlinear and thus may be difficult to cast in terms familiar from statistical analysis of the general linear model. However, it seems obvious that a generalized analysis of nonlinear models would permit a similar identification. One role of the specification of ρ could be to guide the direction that such nonlinear statistical analysis takes.

[3] It would be possible to extend much of the analysis to deal with the case of a near-riskless asset, such as a bond with some probability of default. Letting the return vector for the bond be denoted $\mathbf{g}$, all that is needed is to replace $\mathbf{1}$ by $\mathbf{g}$ in the definition of invariance.

Example 3. Denote

$$\mathbf{y}^* = \sum_{j=2}^{N} \alpha_j^* \mathbf{y}^j$$

and

$$\mathbf{y} = \sum_{j=2}^{N} \alpha_j \mathbf{y}^j.$$

Then for the MAD class of preferences,

$$r(\boldsymbol{\alpha}) - r = \frac{|\mathbf{y} - E(\mathbf{y})|\mathbf{1}}{|\mathbf{y}^* - E(\mathbf{y}^*)|\mathbf{1}} [r(\boldsymbol{\alpha}^*) - r].$$

6. Concluding comments

The tractability of the mean-standard deviation model of choice under risk has made it the standard tool of applied financial analysis. However, the variance is not a particularly attractive index of the riskiness of financial assets, particularly in view of the large body of analysis showing that investors prefer distributions of returns that are skewed to the right. In this paper, the crucial property of the standard deviation as an index of risk, namely its independence of location and scale, has been abstracted and systematically analyzed. The starting point has been the observation that this property is analogous in many important respects to the more restrictive property of constant risk aversion.

The tractability of invariant risk preferences naturally raises the issue of whether such preferences are consistent with EU theory. The additive-separability property of EU preferences has proved useful in many contexts, and the normative appeal of the EU axioms remains strong, despite the vigorous criticism that has been mounted for at least the past two decades. As previous work has shown, however, EU preferences are consistent with constant risk aversion only in the trivial case of risk neutrality. In this paper, it has been shown that EU preferences are invariant only in the case where the utility function is quadratic, that is, where EU theory coincides with mean-standard deviation analysis. The quadratic utility model has unappealing features including increasing absolute risk aversion. Thus, to the extent that invariant preferences are useful, EU theory may prove unduly restrictive.

The analysis presented here shows that any index of riskiness, independent of scale and location, is associated with a flexible family of generalized EU preferences with invariant risk attitudes. The analysis presented here is also applicable, with only modest modifications, to problems of inequality measurement. In particular, we have shown that to any translation invariant, positively linearly homogeneous risk index there corresponds a class of well-behaved certainty-equivalent representations of preferences. Similarly, if we interpret the risk index as an inequality index, there exists a corresponding class of social welfare functions dual to K^*.

In summary, the assumption of invariant risk preferences yields many of the desirable properties of CARA and CRRA preferences, without the unappealing restrictions that arise from the combination of CARA and CRRA.

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Appendix A

Proof of Result 1. (ii) and (iii) $\Rightarrow$ (i): For any $\mathbf{y}, \delta$, choose λ to satisfy $\frac{(1-\lambda)}{\lambda} E(\mathbf{y}) = \delta$ and let $\mathbf{y}' = \mathbf{y}/\lambda$. Note that CRRA requires $\lambda e(\mathbf{y}') = e(\lambda \mathbf{y}') = e(\mathbf{y})$. By linearity on equal mean sets

$$\begin{aligned} e(\mathbf{y} + \delta \mathbf{1}) &= e(\lambda \mathbf{y}' + (1-\lambda) E(\mathbf{y}')\mathbf{1}) \\ &= \lambda e(\mathbf{y}') + (1-\lambda) E(\mathbf{y}') \quad \text{by linearity on equal mean sets} \\ &= e(\lambda \mathbf{y}') + (1-\lambda) E(\mathbf{y}') \quad \text{by CRRA} \\ &= e(\mathbf{y}) + \delta, \end{aligned}$$

which establishes CARA.

(i) and (iii) $\Rightarrow$ (ii): For any $\mathbf{y}, t > 0$,

$$\begin{aligned} te(\mathbf{y}) + (1-t)E(\mathbf{y}) &= e(t\mathbf{y} + (1-t)E(\mathbf{y})\mathbf{1}) \quad \text{by linearity on equal mean sets} \\ &= e(t\mathbf{y}) + (1-t)E(\mathbf{y}) \quad \text{by CARA} \end{aligned}$$

and the conclusion follows immediately.

(i) and (ii) $\Rightarrow$ (iii): For any $\mathbf{y}, \lambda > 0$

$$\begin{aligned} e(\lambda \mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}) &= e(\lambda \mathbf{y}) + (1-\lambda)E(\mathbf{y}) \quad \text{by CARA} \\ &= \lambda e(\mathbf{y}) + (1-\lambda)E(\mathbf{y}) \quad \text{by CRRA.} \end{aligned}$$

Proof of Result 2. (ii) and (iii) $\Rightarrow$ (i): For any δ and $\mathbf{y}, \mathbf{y}' \in M(\mu)$ such that $e(\mathbf{y}) \geqslant e(\mathbf{y}')$, choose λ to satisfy $\frac{(1-\lambda)}{\lambda}\mu = \delta$. Now

$$e(\mathbf{y}/\lambda) \geqslant e(\mathbf{y}'/\lambda) \quad \text{by radial invariance,}$$

so

$$e(\mathbf{y}+\delta\mathbf{1}) = e\Big(\lambda(\mathbf{y}/\lambda)+(1-\lambda)\frac{\mu}{\lambda}\mathbf{1}\Big)$$
$$\geqslant e\Big(\lambda(\mathbf{y}'/\lambda)+(1-\lambda)\frac{\mu}{\lambda}\mathbf{1}\Big) \quad \text{by multiplicative spread invariance}$$
$$= e(\mathbf{y}'+\delta\mathbf{1}) \quad \text{by construction.}$$

(i) and (iii) $\Rightarrow$ (ii): For any $t \geqslant 0$ and $\mathbf{y}, \mathbf{y}' \in M(\mu)$ such that $e(\mathbf{y}) \geqslant e(\mathbf{y}')$, let $\delta = \left(\frac{1}{1-t}\right)\mu$. Now

$$e(\mathbf{y}+\delta\mathbf{1}) \geqslant \mathbf{e}(\mathbf{y}'+\delta\mathbf{1}) \quad \text{by translation invariance,}$$

so

$$e(t\mathbf{y}) = e(t(\mathbf{y}+\delta\mathbf{1})+(1-t)(\mu+\delta)\mathbf{1})$$
$$\geqslant e(t(\mathbf{y}'+\delta\mathbf{1})+(1-t)(\mu+\delta)\mathbf{1}) \quad \text{by multiplicative spread invariance}$$
$$= e(t\mathbf{y}') \quad \text{by construction.}$$

(i) and (ii) $\Rightarrow$ (iii): For any $\lambda \geqslant 0$ and $\mathbf{y}, \mathbf{y}' \in M(\mu)$ such that $e(\mathbf{y}) \geqslant e(\mathbf{y}')$, let $\delta = (1-\lambda)E(\mathbf{y})$. Now

$$e(\lambda\mathbf{y}) \geqslant e(\boldsymbol{\lambda}\mathbf{y}') \quad \text{by radial invariance.}$$

Hence

$$e(\lambda\mathbf{y}+(1-\lambda)E(\mathbf{y})\mathbf{1}) = e(\lambda\mathbf{y}+\delta\mathbf{1})$$
$$\geqslant e(\lambda\mathbf{y}'+\delta\mathbf{1}) \quad \text{by translation invariance}$$
$$= e(\lambda\mathbf{y}'+(1-\lambda)E(\mathbf{y}')\mathbf{1}) \quad \text{by construction.}$$

Proof of Result 3. Our strategy for this proof is to establish a property that all differentiable and invariant EU functions must satisfy. Then, we show that this condition is satisfied if and only if the EU functional is quadratic.

Claim 1. *Suppose the risk-ordering is invariant, that is, translation invariant, radially invariant and multiplicative spread invariant. If*

$$E(\mathbf{y}) = E(\mathbf{y}') = \mu$$

and

$$E(u(\mathbf{y})) = E(u(\mathbf{y}')),$$

where $u(\mathbf{y})(s) = u(y(s))$, *then*

$$E((\mathbf{y}-\mu\mathbf{1})\nabla u(\mathbf{y})) = E((\mathbf{y}'-\mu\mathbf{1})\nabla u(\mathbf{y}')),$$

where $\nabla u(\mathbf{y})(s) = u'(y(s))$.

Proof. We have

$$E(u(\lambda\mathbf{y}+(1-\lambda)\mu\mathbf{1})) = E(u(\lambda\mathbf{y}'+(1-\lambda)\mu\mathbf{1}))$$

for all $\lambda > 0$ and hence

$$\frac{\partial}{\partial \lambda} E(u(\lambda \mathbf{y} + (1-\lambda)\mu \mathbf{1})) = \frac{\partial}{\partial \lambda} E(u(\lambda \mathbf{y}' + (1-\lambda)\mu \mathbf{1})).$$

Evaluating this expression at $\lambda = 1$ gives

$$E((\mathbf{y} - \mu\mathbf{1})\nabla u(\mathbf{y})) = E((\mathbf{y}' - \mu\mathbf{1})\nabla u(\mathbf{y}')), \tag{A.1}$$

as claimed. □

Our next step is to show that (A.1) is satisfied if and only if u is quadratic. Taking Taylor-series expansions around $y = \mu$, we have, respectively,

$$u(y) = u(\mu) + u'(\mu)(y-\mu) + \tfrac{1}{2}u''(\mu)(y-\mu)^2 + \cdots$$

and

$$u'(y) = u'(\mu) + u''(\mu)(y-\mu) + \tfrac{1}{2}u'''(\mu)(y-\mu)^2 + \cdots.$$

Therefore,

$$E[u(\mathbf{y})] = u(\mu) + \tfrac{1}{2}u''(\mu)E[(\mathbf{y} - \mu\mathbf{1})^2] + \cdots$$

and

$$\begin{aligned} E((\mathbf{y} - \mu\mathbf{1})\nabla u(\mathbf{y})) &= u'(\mu)E((\mathbf{y} - \mu\mathbf{1})) + u''(\mu)E((\mathbf{y} - \mu\mathbf{1})^2) \\ &\quad + \tfrac{1}{2}u'''(\mu)E((\mathbf{y} - \mu\mathbf{1})^3) + \cdots && \text{(A.2)} \\ &= u''(\mu)\sigma_y^2 + \tfrac{1}{2}u'''(\mu)E((\mathbf{y} - \mu\mathbf{1})^3) + \cdots . && \text{(A.3)} \end{aligned}$$

Now observe that if u is quadratic, then

$$E((\mathbf{y} - \mu\mathbf{1})\nabla u(\mathbf{y})) = u''(\mu)\sigma_y^2.$$

Moreover, the conditions that

$$E(\mathbf{y}) = E(\mathbf{y}') = \mu$$

and

$$E(u(\mathbf{y})) = E(u(\mathbf{y}'))$$

imply $\sigma_y^2 = \sigma_{y'}^2$. Hence,

$$\begin{aligned} E((\mathbf{y} - \mu\mathbf{1})\nabla u(\mathbf{y})) &= u''(\mu)\sigma_y^2 \\ &= u''(\mu)\sigma_{y'}^2 \\ &= E((\mathbf{y}' - \mu\mathbf{1})\nabla u(\mathbf{y}')), \end{aligned}$$

as required by an invariant differentiable EU preference structure. □

The main result: Eq. (A.1) is satisfied for all $\mathbf{y}, \mathbf{y}'$ such that

$$E(\mathbf{y}) = E(\mathbf{y}') = \mu$$

and

$$E(u(\mathbf{y})) = E(u(\mathbf{y}')),$$

if and only if u is quadratic.

Proof. We have already proved sufficiency. Under the stated conditions, for any $\lambda \in (0, 1)$,

$$E(\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1}) = E(\lambda\mathbf{y}' + (1-\lambda)\mu\mathbf{1}) = \mu$$

and

$$E(u(\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1})) = E(u(\lambda\mathbf{y}' + (1-\lambda)\mu\mathbf{1}))$$

by invariance. Note that

$$\mathrm{var}(\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1}) = \lambda^2\sigma_y^2,$$

$$\mathrm{var}(\lambda\mathbf{y}' + (1-\lambda)\mu\mathbf{1}) = \lambda^2\sigma_{y'}^2.$$

We may therefore rewrite the Taylor expansion in (A.3) for arbitrary λ:

$$E(((\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1}) - \mu)\nabla u(\lambda\mathbf{y} + (1-\lambda)\mu\mathbf{1})) \\ = \lambda^2 u''(\mu)\sigma_y^2 + \tfrac{1}{2}\lambda^3 u'''(\mu)E((\mathbf{y} - \mu\mathbf{1})^3) + \cdots \,. \quad \text{(A.4)}$$

For λ suitably close to zero, this expansion will be dominated by the lowest-order term, namely $\lambda^2 u''(\mu)\sigma_y^2$. Hence

$$E((\mathbf{y} - \mu\mathbf{1})\nabla u(\mathbf{y})) = E((\mathbf{y}' - \mu\mathbf{1})\nabla u(\mathbf{y}'))$$

only if

$$\lambda^2 u''(\mu)\sigma_y^2 = \lambda^2 u''(\mu)\sigma_{y'}^2,$$

that is, only if

$$\sigma_y^2 = \sigma_{y'}^2$$

whenever $E(\mathbf{y}) = E(\mathbf{y}') = \mu$ and $E(u(\mathbf{y})) = E(u(\mathbf{y}'))$.

This is true only if u is quadratic. □

Proof of Lemma 1. Define the gauge function of the closed convex set A by

$$d(\mathbf{y}, A) = \inf\{t > 0 : \mathbf{y} \in tA\},$$

if there is a t such that $\mathbf{y} \in tA$, and ∞ otherwise. d is positively linearly homogeneous and subadditive (sublinear) in $\mathbf{y}$ [2, Lemma 5.36]. It is lower semicontinuous in $\mathbf{y}$ if and only if A contains zero [2, Theorem 5.39]. By construction,

$$r(\mathbf{y}, \mathbf{y}') = d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}).$$

If preferences are quasi-concave, $K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}$ is a translate of a closed convex set and hence closed and convex. If preferences are risk-averse, $E(\mathbf{y}')\mathbf{1} \in K(\mathbf{y}')$, which implies that $K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}$ is a closed convex set containing the origin. Thus, $r(\mathbf{y}, \mathbf{y}')$ is lower semicontinuous, positively linearly homogeneous, and subadditive in $\mathbf{y} - E(\mathbf{y})\mathbf{1}$. The last implies that $r(\mathbf{y}, \mathbf{y}')$ is lower semicontinuous in $\mathbf{y}$. By subadditivity,

$$\begin{aligned} r(\mathbf{y} + \mathbf{y}^*, \mathbf{y}') &= d(\mathbf{y} + \mathbf{y}^* - E(\mathbf{y} + \mathbf{y}^*)\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) \\ &= d(\mathbf{y} - E(\mathbf{y})\mathbf{1} + \mathbf{y}^* - E(\mathbf{y}^*)\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) \\ &\leqslant d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) + d(\mathbf{y}^* - E(\mathbf{y}^*)\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) \\ &= r(\mathbf{y}, \mathbf{y}') + r(\mathbf{y}^*, \mathbf{y}') \end{aligned}$$

and

$$\begin{aligned} r(t\mathbf{y}, \mathbf{y}') &= d(t\mathbf{y} - tE(\mathbf{y}), K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) \\ &= td(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}). \end{aligned}$$

By construction, $r(\mathbf{y}, \mathbf{y}') \geqslant 0$.

If preferences are strictly risk-averse, then, for any $\mathbf{y} \in M(0), r(\mathbf{y}, \mathbf{y}') = 0$ if and only if $\mathbf{y} = \mathbf{0}$. Sufficiency is obvious. For necessity, observe that if $\mathbf{y} \in M(0)$, but $\mathbf{y} \neq \mathbf{0}$, strict risk aversion implies that

$$e(\mathbf{y}) < E(\mathbf{y}) = 0 = \lim_{t \to 0} e(t(\mathbf{y}' - E(\mathbf{y}')\mathbf{1}))$$

and hence $r(\mathbf{y}, \mathbf{y}') > 0$.

Proof of Result 4. By the proof of Lemma 1, $d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1})$ is a positively linearly homogeneous, subadditive, and semicontinuous function of $\mathbf{y} - E(\mathbf{y})\mathbf{1}$. Thus, by the polarity between gauge and support functionals, it can be recognized as a support functional of the form [2, Theorem 5.104]

$$d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) = \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - \mathrm{E}(\mathbf{y})\mathbf{1}) : \mathbf{p} \in K^*\},$$

where

$$K^* = \{\mathbf{p} : d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) \geqslant \mathbf{p}(\mathbf{y} - \mathrm{E}(\mathbf{y})\mathbf{1}) \text{ for all } \mathbf{y} - E(\mathbf{y})\mathbf{1}\}.$$

K^* is closed and convex by the lower semicontinuity and sublinearity (positive linear homogeneity and subadditivity) of d. Since $d \geqslant 0$, K^* always contains the origin.

Proof of Result 5. *Sufficiency*: Suppose $\mathbf{y}, \mathbf{y}' \in M(\mu)$ and $e(\mathbf{y}) \geqslant e(\mathbf{y}')$. Then, under the stated hypothesis

$$\phi(E(\mathbf{y}), \rho(\mathbf{y})) \geqslant \phi(E(\mathbf{y}'), \rho(\mathbf{y}'))$$

and hence, since $E(\mathbf{y}) = E(\mathbf{y}') = \mu$, $\rho(\mathbf{y}) \leqslant \rho(\mathbf{y}')$. Hence,

$$\begin{aligned} \phi(E(\mathbf{y} + \delta\mathbf{1}), \rho(\mathbf{y} + \boldsymbol{\delta}\mathbf{1})) &= \phi(E(\mathbf{y}) + \delta, \rho(\mathbf{y})) \\ &\geqslant \phi(E(\mathbf{y}') + \delta, \rho(\mathbf{y}')) \\ &= \phi(E(\mathbf{y}' + \delta\mathbf{1}), \rho(\mathbf{y}' + \delta\mathbf{1})). \end{aligned}$$

Necessity: Suppose $\mathbf{y}, \mathbf{y}' \in M(\mu)$, then $e(\mathbf{y}) \geqslant e(\mathbf{y}')$ if and only if $\mathbf{y} \in K(\mathbf{y}')$. Hence, $e(\mathbf{y}) \geqslant e(\mathbf{y}')$ if and only if $d(\mathbf{y} - E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) = r(\mathbf{y}; \mathbf{y}') \leqslant 1$. Because

$\mathbf{y}' \in I(e(\mathbf{y}'))$, $d(\mathbf{y}' - E(\mathbf{y}')\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) = r(\mathbf{y}'; \mathbf{y}') = 1$, and thus $\rho(\mathbf{y}) \leqslant \rho(\mathbf{y}')$. For arbitrary $\mathbf{y}$ with $e(\mathbf{y}) > 0$, consider

$$\begin{aligned} d(e(\mathbf{y})\mathbf{y} - e(\mathbf{y})E(\mathbf{y})\mathbf{1}, K(\mathbf{y}') - E(\mathbf{y}')\mathbf{1}) &= r(e(\mathbf{y})\mathbf{y}; \mathbf{y}') \\ &= e(\mathbf{y}) r(\mathbf{y}; \mathbf{y}') \\ &= e(\mathbf{y})\rho(\mathbf{y}). \end{aligned}$$

The properties of ϕ now follow. For $\delta > 0$

$$\begin{aligned} e(\mathbf{y} + \delta\mathbf{1}) &= \phi(E(\mathbf{y}) + \delta, \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}) \\ &\geqslant e(\mathbf{y}) \\ &= \phi(E(\mathbf{y}), \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}), \end{aligned}$$

which establishes that ϕ is nondecreasing in it first argument. The agreement property implies

$$\begin{aligned} e(c\mathbf{1}) &= \phi\left(c, \sup_{\mathbf{p}} \{\mathbf{p0} \ : \ \mathbf{p} \in K^*\}\right) \\ &= \phi(c, 0) \\ &= c. \end{aligned}$$

Finally, under risk aversion and quasi-concavity for $(0 \leqslant \lambda \leqslant 1)$,

$$\begin{aligned} e(\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}) &= \phi\left(E(\mathbf{y}), \sup_{\mathbf{p}} \{\mathbf{p}(\lambda\mathbf{y} + (1-\lambda)E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}\right) \\ &= \phi\left(E(\mathbf{y}), \sup_{\mathbf{p}} \{\mathbf{p}(\lambda\mathbf{y} - \lambda E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}\right) \\ &= \phi\left(E(\mathbf{y}), \lambda \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}\right) \\ &\leqslant \min\{e(\mathbf{y}), E(\mathbf{y})\} \\ &= e(\mathbf{y}) \\ &= \phi\left(E(\mathbf{y}), \sup_{\mathbf{p}} \{\mathbf{p}(\mathbf{y} - E(\mathbf{y})\mathbf{1}) : \ \mathbf{p} \in K^*\}\right), \end{aligned}$$

which establishes that ϕ is not increasing in its second argument.

Proof of Result 6. Sufficiency is trivial. For necessity, observe that under constant absolute risk aversion

$$e(\mathbf{y} + \delta\mathbf{1}) = e(\mathbf{y}) + \delta$$

and hence

$$\phi(\mu + \delta, \rho) = \phi(\mu, \rho) + \delta, \quad \forall \mu, \rho, \delta.$$

Setting $\delta = -\mu$ then gives

$$\phi(\mu, \rho) = \phi(0, \rho) + \mu.$$

Now by CRRA

$$e(t\mathbf{y}) = te(\mathbf{y}),$$

whence

$$\phi(tE(\mathbf{y}), t\rho(\mathbf{y})) = t\phi(E(\mathbf{y}), \rho(\mathbf{y}))$$

and thus

$$\phi(t\mu, t\rho) = t\phi(\mu, \rho),$$

which implies $\phi(0, \rho) = \rho\phi(0, 1)$ and hence

$$\phi(\mu, \rho) = \rho\phi(0, 1) + \mu$$

as required.

Proof of Result 7. By the decomposition in the text, the second-stage problem determines the optimal risk index as

$$\tilde{\rho} \in \arg\max_{\rho} \{\phi(Wr + \rho V(\hat{\mathbf{Y}}, 1), \rho)\}.$$

Let

$$\tilde{\boldsymbol{\alpha}}_{-1} \in \arg\max_{\boldsymbol{\alpha}_{-1}} \left\{ E\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right) - r\sum_{j=2}^{N} \alpha_j : \rho\left(\sum_{j=2}^{N} \alpha_j \mathbf{y}^j\right) \leqslant 1 \right\}.$$

As shown in the body of the text, for any ρ, $\rho\tilde{\boldsymbol{\alpha}}_{-1}$ is a solution to $V(\hat{\mathbf{Y}}, \rho)$ and therefore, if ρ is optimal for a given W, then $\rho\tilde{\boldsymbol{\alpha}}_{-1}$ is optimal. By the budget constraint, therefore, $\alpha_1 = W - \rho\sum_{j=2}^{N} \tilde{\alpha}_j$.

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