Risk and Uncertainty Program

TITLE: Minimal Conditions for Parametric Continuity of a Utility Representation

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Working Paper: R15_1

Minimal conditions for parametric continuity of a utility representation[*]

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Abstract

When sufficiently small perturbations of parameters preserve strict preference for one alternative over another, dependence on the parameters is continuous. We characterise this property with a utility function over alternatives that depends continuously on the parameter. The class of parameter spaces such that this form of representation is guaranteed to exist is also characterised. When the parameters are beliefs, these results have implications for robust portfolio choice, Bayesian games and psychological games. When alternatives are discrete, the representation is jointly continuous, and an extension of Berge's theorem of the maximum, yields a continuous value function. We apply this result to generalise a standard consumer choice problem: where parameters are price-wealth vectors. When the parameter space is lexicographically ordered, a novel application to reference-dependent preferences is possible.

1   Introduction

It is often natural to assume that strict preference for one alternative over another is preserved when a parameter upon which preferences depend is perturbed. We refer to this property as *continuous parameter dependence* of preferences. The following quote provides the behavioural motivation for this assumption.

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> When processing sensory input, it is of vital importance for the
> neural systems to be able to discriminate a novel stimulus from
> the background of redundant, unimportant signals.
>
> (Mejias et al., 2012)

In other words, in the absence of robust preference, errors would compound, and we would be unable to perform many of our day-to-day activities. A similar point is made by Rolls and Deco (2010), and these views may be traced back to von Neumann (1976).

In this paper, continuous parameter dependence is captured through a condition or axiom on preferences that vary across a parameter space. Essential to the one-shot decision framework we have in mind, is the assumption that the decision maker's preferences are a ranking of the alternatives given each parameter. Beyond this, the intended meaning of the term "parameter" in this paper is broad as possible and this is reflected in the applications and examples we provide following the exposition of the model and results.

In the main theorem of this paper, we identify minimal conditions such that continuous parameter dependence is characterised by utility representation that is continuous in the parameter. The sense in which the conditions are minimal are as follows. First, the axioms on preferences are necessary and sufficient for the representation. Second, when the parameter space fails to satisfy the conditions of this theorem, there always exist preferences with no continuous representation even though they vary continuously with the parameter.

The concept of parametric continuity demands conditions on the nature of the open sets of the parameter space. That is, conditions that are topological in nature. In this paper we show that in order to guarantee the existence of a utility representation of preferences that is continuous in the parameter, the minimal requirement is that the parameter space is both *perfect* and *normal*. Respectively, this means that every closed set (this includes singletons) is equal to the intersection of a countable collection of open sets; and every disjoint pair of closed sets can be separated by a disjoint pair of open sets. The requirement that the space is normal is obviously relevant for perturbations, however the key to parametric continuity of the representation is that the space is also perfect. The class of perfectly normal spaces includes, but is not limited to, spaces that are metrizable. To our knowledge, there are no other utility representation theorems for a parameter space that is perfectly normal.

Throughout, we assume that the set of alternatives is countable and independent of the parameter space. Thus, no topological assumptions regarding the alternatives are necessary for our main theorem. However, when the set

of alternatives is discrete (so that every element is isolated), our conditions are also necessary and sufficient for the representation to be jointly continuous on pairs of alternatives and parameters. This ensures that, for the discrete case, our representation generalises existing results from the existing literature (Caterino, Ceppitelli, and Maccarino, 2009; Levin, 1983) on jointly continuous utility which provide only sufficient conditions for such a representation of preferences. The survey of this literature appears in subsection 3.2.

Joint continuity is useful for deriving a continuous value function (e.g. a continuous indirect utility function in consumer theory). In particular it is a premise of Berge's theorem of the maximum which provides sufficient conditions for such a value function to exist. The other premise for Berge's theorem is that the feasible set of alternatives the decision maker faces varies continuously with the parameter. Indeed, in the standard consumer choice setting, the budget correspondence is continuous. When the set of alternatives is discrete, this latter requirement is too strong. It implies that the feasible set of alternatives is constant across the parameter space. For the case of a discete set of alternatives, we extend Berge's theorem so that the constraint set only varies upper hemicontinuously. The result is a value function that is continuous and a choice correspondence that is upper hemicontinuous (on a perfectly normal parameter space). To our knowledge, this result is novel and related to the literature on envelope theorems with discrete alternatives (Milgrom and Segal, 2002; Sah and Zhao, 1998).

Our first application builds on this latter result. In subsection 4.1, we consider a standard consumer choice setting, where alternatives are commodities and the parameter space is the product of price-wealth pairs with a further space of parameters. We introduce the latter augmentation with a view to demonstrating the potential for applications with a general specification of experimental frames, reference points, unawareness, beliefs and types. In such settings our results demonstrate that it is not necessary to require the parameter space is metrizable, but if it is, then parametric continuity is guaranteed.

When the parameter space consists of beliefs (probability measures) on a state space, the topology on the set states is also important for parametric continuity of the utility and corresponding value function. This is particularly true when the set of states is infinite. In particular for a compact set of states: the set of states is metrizable if and only if the set of beliefs are metrizable; but when the set of states is compact, but not perfectly normal, the set of beliefs also fails to be perfectly normal. This implies that even when preferences depend continuously on beliefs, the corresponding utility and value functions may be discontinuous.

In our second application, we use this latter result in combination with our main results to highlight some immediate implications for topologies on types (Dekel, Fudenberg, and Morris, 2006; Mertens and Zamir, 1985), psychological games (Geanakoplos, Pearce, and Stacchetti, 1989) and utility maximisation in the context of a game (Gilboa and Schmeidler, 2003). A brief discussion of these implications is provided in subsection 4.2.

Our third application is to belief dependence, where the parameter space is a set of probability measures on a set of states. The canonical example takes each state to be a path across time that represents portfolio value. In this setting, it is common practice for investment advisors to present their clients with perturbed paths, so as to check that their portfolio choice is robust. We confirm that, provided the set of paths is metrizable, this is tantamount to perturbing beliefs, and moreover perturbing the utility representation is equivalent to perturbing strict preference. However, once again, when the set of states is not perfectly normal, the set of beliefs is not perfectly normal, and parametric perturbations of the utility or value function may be misleading. To demonstate how easily this might arise, we outline two approaches to constructing the same measure that governs a Brownian motion. The first yields a metrizable space of beliefs whereas the second yields one that is not perfectly normal. This highlights a subtle issue that may often go unnoticed in model specifications.

Our fourth and final application is a simple, two alternative choice problem. Here, the parameters are reference points. The innovation is to allow for a lexicographic ordering of reference points. This yields an example that is uniquely suited to the present model, for there are lexicographically ordered sets that are perfectly normal, but not metrizable.

The next section introduces the model along with preliminary observations. The main theorem and related results appear in section 3. Following this, the applications are presented in section 4. Following the summary in section 5, the proof of the main theorem is presented in appendix A1. All remaining proofs appear in appendix A2.

2 Model

In this section we first present the basic model of preferences that depend on a parameter regardless of issues relating to continuity. We then define continuous parameter dependence and describe basic topological conditions on the parameter space. Following this, more specific conditions on the parameter space are defined and explored using examples.

2.1 Preferences indexed by a general set of parameters

Let A denote a nonempty set of *alternatives*. Let X denote a nonempty set. We refer to an element of X as a *parameter*. Motivated by our interest in robust strict preference, we take statements of strict preference as primitive. For each x in X, $a \prec_x b$ denotes the statement "at x, alternative b is strictly preferred to alternative a". For each x in X, $\prec_x$ is the binary relation A that summarises all such preference statements. As such $\prec_x$ is a subset of $A \times A$, and is referred to as *preferences at x* or *given x*. For alternatives a and b such that neither $a \prec_x b$, nor $b \prec_x a$, we write $a \sim_x b$.

The shorthand $\{\prec_x\}_{x \in X}$ denotes the collection "$\prec_x$ such that x belongs to X", and is the primitive object we refer to as *preferences*. Preferences are *parameter-free* whenever X is a singleton, otherwise, they are *parameter-dependent*. Thus, we do not require that each x determines a unique $\prec_x$. Indeed, typically, the mapping $x \mapsto \prec_x$ is many-to-one. The term *parameter dependence* will be used without reference to preferences when no possible confusion might arise. Similarly, we henceforth refer to "the decision maker" as Val.

Representing parameter dependence By a *representation* of preferences, we mean a function of the form $U : A \times X \to \mathbb{R}$ such that, for every x in X, and every $a, b \in A$, $a \prec_x b$ if and only if $U(a, x) < U(b, x)$. That is, for each x in X, there exists a *utility function* $u = U(\cdot, x) : A \to \mathbb{R}$ that represents preferences at x in the usual sense. By the properties of $<$ on $\mathbb{R}$, it is straightforward to show that if preferences have a representation, then they satisfy the following two conditions.

Axiom Asy. *If $a, b \in A$, then, for every $x \in X$, $a \prec_x b$ implies not $b \prec_x a$.*

Axiom NT. *If $a, b, c \in A$, then, for every $x \in X$, $a \prec_x b$ implies $c \prec_x b$ or $a \prec_x c$.*

In turn, when A is countable, *asymmetry* (Asy) and *negative transitivity* (NT) are standard sufficient conditions for the existence of a utility function at each $x \in X$. That is, sufficient for a representation of $\{\prec_x\}_{x \in X}$ in this case. With minor modifications, this result is due to Cantor (1895). More generally, (Asy) and (NT) are equivalent to assuming the weak preference relation $\precsim_x = \prec_x \cup \sim_x$ is complete and transitive for every $x \in X$.[1] With a view to finding the weakest conditions for parametric continuity, we henceforth assume that A is countable unless otherwise stated.

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[1] Recall that $\precsim_x$ is complete if, for all a, b and c in A, $a \precsim_x b$ or $b \precsim_x a$ and $\precsim_x$ is transitive if $a \precsim_x b \precsim_x c$ implies $a \precsim_x c$. See Fishburn (1979) for more on the relationships between these and other conditions.

2.2 Continuous parameter dependence

If preferences are such that, for each $a, b \in A$ such that $a \prec_x b$, there exists an "open neighbourhood" N of x in X such that $a \prec_y b$ for every other y in N, then we say that *parameter dependence is continuous at x*. Note that whenever $\prec_x = \varnothing$, this condition says nothing about preferences. The notion of an open set of X is only well-defined once a topology on X is identified.

The topology on X provides a constraint on the perturbations that are allowed. Recall that a *topology* on X is any collection τ of subsets G of X such that τ is closed under finite intersections and arbitrary unions. Usually we will suppress reference to τ and simply call X a topological space. Thus, by "G is open" we mean $G \in \tau$ and by "F is closed" we mean that for some $G \in \tau$, F is equal to the complement $X - G$ of G.

Recall that a neighbourhood of x is some subset $N \subseteq X$ such that $x \in N$ and $G \subseteq N$ for some open set G. We will maintain the assumption that, for each $x \in X$, the singleton set $\{x\}$ is equal to the intersection of all closed neighbourhoods of x. This is equivalent to the assumption that X is a Hausdorff space: every distinct pair $x, y \in X$ gives rise to a disjoint pair of open neighbourhoods N_x and N_y of x and y respectively.

Suppose that for some $y \in X$, parameter dependence is discontinuous at y. Then, for some $a, b \in A$ satisfying $a \prec_y b$ we have: for every open neighbourhood N of y, there exists $x \in N$ such that $b \precsim_x a$. (This follows directly from the definition of $\prec_x$ and $\sim_x$.) Thus, discontinuous parameter dependence for some $y \in X$ is equivalent to the set $\{x : a \prec_x b\}$ failing to be open for some $a, b \in A$. On the other hand, when parameter dependence is continuous at x for every $x \in X$, we have

Axiom CD. *For every $a, b \in A$, the set $\{x : a \prec_x b\}$ is open in X.*

This axiom appears in Gilboa and Schmeidler (1997, 2003). Indeed the authors also assume completeness and transitivity of weak preference at each x. Whilst we identify minimal conditions for parametric continuity of the representation, Gilboa and Schmeidler (1997, 2003) impose further axioms and obtain a representation that is linear in the parameter.

When A is discrete (every subset is both open and closed), (CD) is equivalent to assuming that the correspondence $x \mapsto \prec_x$ is lower hemicontinuous (l.h.c.). (This means that, for every open $G \subseteq A \times A$ the set $\{x : \prec_x \cap G \neq \varnothing\}$ is open.) For example, because A is discrete, $G = \{(a, b)\}$ is open and $\{x : \prec_x \cap G \neq \varnothing\} = \{x : a \prec_x b\}$.

Characterising continuous parameter dependence For any function $U : A \times X \to \mathbb{R}$, we will say that U is *continuous at x* whenever the function

$U(a, \cdot) : X \to \mathbb{R}$ is continuous at x for every $a \in A$.

Lemma 2.1. *Let $U : A \times X \to \mathbb{R}$ be a representation of preferences $\{\prec_x\}_{x \in X}$. If U is continuous at x, then parameter dependence is continuous at x.*

Proof of lemma 2.1. Let G be the set of points $x \in X$ such that parameter dependence is continuous at x, and let H be the set of points $x \in X$ such that U is continuous at x. We will show that $H \subseteq G$. Suppose parameter dependence is discontinuous at x (so that $x \in X - G$). Then, for some $a, b \in A$ satisfying $a \prec_x b$: every open neighbourhood N of x, there exists $y \in N$ such that $b \precsim_x a$. Consider the collection $\{N_\nu\}_{\nu \in D}$ of all neighbourhoods of x partially ordered by the inclusion (subset) relation $\subseteq$. Then we may take D to be a directed set that generates a net $(y_\nu)_{\nu \in D}$. This net is such that: $b \precsim_{y_\nu} a$ for every $\nu \in D$; and it converges to x. (This latter fact follows from our assumption that X is a Hausdorff set.) Now since U is a representation, this means that $U(a, x) - U(b, x) < 0$ and for every ν, $U(a, y_\nu) - U(b, y_\nu) \leqslant 0$. Then zero is an upper bound for the latter set of points. Thus U is discontinuous at x, and so $x \in X - H$. $\square$

When U is continuous at x for every $x \in X$, we say U is *continuous in the parameter* or that it satisfies *parametric continuity*. Lemma 2.1 immediately implies that if the representation U satisfies parametric continuity, then (CD) holds. The following statement is significantly weaker than the converse of lemma 2.1, yet, even for two alternatives, it requires further conditions on the parameter space.

> *Let* (Asy), (NT) *and* (CD) *hold. Then there exists a representation that is continuous in the parameter.*

If this statement holds, then U *characterises* continuous parameter dependence. Our purpose for the remainder of this section is to define and explore the minimal conditions on the parameter space such that it does.

2.3 Perfectly normal parameter spaces

A (topological) space X is *perfectly normal* if it is both normal and perfect. A space X is *normal*, if every pair of closed sets E, F can be separated. That is, there exist open sets G and H in X such that $E \subseteq G$ and $F \subseteq H$ and, moreover, $G \cap H$ is empty. A space is *perfect* if every closed set is the intersection of countably many open sets in X.

The following example shows precisely how this property is used to construct a utility representation for the case where there are just two alternatives. The main theorem extends this to the countable case using an

equivalent definition of perfect normality (Michael's selection theorem) that we present below.

Example 2.1. *Let $A = \{a, b\}$, and suppose that Y_{ab} denotes the nonempty, open set $\{x : a \prec_x b\}$. If Val's preferences satisfy (CD), then the set Y_{ab} is open and $X - Y_{ab}$ is closed. Similarly, the set $Y_{ba} = \{x : b \prec_x a\}$ is then open and we suppose it is nonempty. If Val's preferences also satisfy (Asy), then Y_{ab} and Y_{ba} are disjoint. Recall that if two sets are open and disjoint, then they are separated sets. That is, neither contains a limit point of the other.*

If X is perfect, the fact that $X - Y_{ba}$ is closed implies that there exists a countable collection $\{G_n\}_{n \in \mathbb{N}}$ of open sets satisfying $\bigcap_1^\infty G_n = X - Y_{ba}$. If X is also normal, then the Urysohn lemma applies because $X - Y_{ba}$ and $X - G_n$ are closed and disjoint. This guarantees the existence of a real-valued continuous function f_n on X such that $f_n(x) = 0$ on $X - Y_{ba}$ and $f_n(x) = 1$ on $X - G_n$, and $0 \leqslant f_n(x) \leqslant 1$ otherwise. Now let $f = \sum_1^\infty 2^{-n} f_n$. As the uniform limit of continuous functions is continuous, f is a continuous, nonnegative function. Moreover, for every $x \in X - Y_{ba}$, since $f_n(x) = 0$ for each n, $f(x) = 0$. For every other x, there exists n such that $x \in X - G_n$, so that $f(x) > 0$.

By the same argument, there exists another continuous nonnegative function g such that $g^{-1}(0) = X - Y_{ab}$. Now let $U(a, \cdot) \equiv 0$ and let

$$U(b, x) = \begin{cases} f(x) & \text{if } x \in Y_{ab}, \\ -g(x) & \text{if } x \in Y_{ba}, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting function $U : A \times X \to \mathbb{R}$ is a utility representation at each x and, moreover, it is continuous in the parameter. As such, it characterises the continuous parameter dependence of Val's preferences.

There are a number of other, equivalent definitions for perfect normality of a space. For the first, we introduce the concept of a zero set. $F \subseteq X$ is a *zero set* provided that $f^{-1}(0) = F$ for some continuous function $f : X \to \mathbb{R}$. Recall that, since $\{0\}$ is closed in $\mathbb{R}$, for any continuous $f : X \to \mathbb{R}$, the set $F = f^{-1}(0)$ is closed in S. That is, the zero sets are always closed. The converse is only true when X is perfectly normal.

Definition. *X is perfectly normal if and only if every closed subset of X is a zero set.*

A third, equivalent definition of perfect normality is provided by the following restatement of Michael's selection theorem (Michael, 1956, Theorem 3.1‴).

Theorem (Good and Stares (2000)). *X is perfectly normal if and only if whenever $g, h : X \to \mathbb{R}$ are upper and lower semi-continuous respectively and $g \leqslant h$, then there is a continuous $f : X \to \mathbb{R}$ such that $g \leqslant f \leqslant h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.*

Examples of perfectly normal spaces   It is not hard to see that a metrizable space is perfectly normal, for instance, in example 2.1, we can take the sets G_n to be explicit ϵ-neighbourhoods of the closed sets that we seek to separate. Indeed, by considering the usual metric $|\cdot|$ on the nonnegative real numbers $\mathbb{R}_+$, the latter is metrizable and hence perfectly normal.

An example of a set that is perfectly normal, compact, but not metrizable is developed in the context of an application to reference dependence in subsection 4.4. This is the product $[0,1] \times_{\text{lex}} \{0,1\}$ of the unit interval $[0,1]$ with the two-element, discrete set $\{0,1\}$ with topology generated by the lexicographic order (where the first dimension is dominant). It is sometimes referred to the *split interval.*

The split interval is important, because all compact, separable ordered spaces are order isomorphic to one of its subsets (Ostaszewski, 1974). Another space that is homeomorphic to the split interval is the set $\mathbb{F}$ of increasing functions on $[0,1]$ with values in $\{0,1\}$ with the topology of pointwise convergence (Vaughan, 1971).[2,3] As a result, the latter is compact and perfectly normal, but not metrizable. As we show in the sequel, these examples of 4.4 serve to distinguish the present model from its complement.

What if the space fails to be perfectly normal?   Whilst axiom (CD) is often well motivated in applications, it is natural to question the importance of representing continuous parameter dependence. That is to say: what if preferences satisfy (Asy), (NT) and (CD), but there is no representation that is continuous in the parameter? The following two alternative example shows, the discontinuities that are essential (present in all representations) are on the boundary of $\{x : a \prec_x b\}$, where preferences switch from strict preference for b over a to indifference. This is precisely where the continuity property of the representation is most useful.

Example 2.2. *Let $A = \{a, b\}$ and suppose Val's preferences $\{\prec_x\}_{x \in X}$ satisfy (Asy), (CD), and that $F = \{x : a \sim_x b\}$ for some closed set $F \subsetneq X$. Suppose that F is not a zero set. That is, there is no continuous, real-valued g on X*

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[2] Recall that two sets are homeomorphic whenever there is a continuous bijection with continuous inverse between them.

[3] In this case, the topology of pointwise convergence is the collection of all unions of finite intersections of sets $W(x, G) = \{f \in \mathbb{F} : f(r) \in G\}$, where $r \in [0, 1]$ and $G \subseteq \{0, 1\}$.

such that $g^{-1}(0) = F$. (See 4.4 for an explicit example of such a set.) Since preferences satisfy (Asy)*, there exists a representation of preferences. Let $U : A \times X \to \mathbb{R}$ be any such representation. Let $f = U(a, \cdot) - U(b, \cdot)$. Then since U is a representation, $f^{-1}(0) = F$. This implies that $U(a, \cdot) - U(b, \cdot)$ is discontinuous. Then since the sum of two continuous functions is continuous, at least one of $U(a, \cdot)$ and $U(b, \cdot)$ is discontinuous.*

Suppose f is discontinuous at x and $f(x) > 0$. By the definition of f and (CD)*, there exists an open neighbourhood N of x such that, $f(y) > 0$ for every $y \in N$. Now $X - N$ is closed and separated from $\{x\}$ which is also closed. By the same argument as in example 2.1, there exists a continuous function $\lambda : X \to [0, 1]$ satisfying $\lambda^{-1}(0) = X - N$ and $\lambda^{-1}(1) = \{x\}$. Continuity of λ is sufficient for $f' = (1 - \lambda)f + \lambda$ to be continuous at x. In turn, f' gives rise to a new representation U' that is also continuous at x.*

The preceding paragraph demonstrates that any discontinuity of U that lies outside F can be removed. This, together with the fact that $U(a, x) - U(b, x) = 0$ for every $x \in F$, implies that the discontinuities of U that cannot be removed lie on the boundary of F. This implies that, for some $x \in F$, every representation $U : X \to \mathbb{R}$ has the property that there exists (x_ν) converging to x such that $U(a, x_\nu) - U(b, x_\nu)$ does not converge to $0 = U(a, x) - U(b, x)$.

The issues that arises when, as in example 2.2, the representation is discontinuous and preferences satisfy (CD), are as follows. First, a comparative statics exercise is difficult, for the modeller cannot rely on closeness to the boundary of $\{x : a \prec_x b\}$ being associated with eventual closeness of $U(a, \cdot) - U(b, \cdot)$ to zero. (Limits can no longer be relied upon for derivatives for instance.) The second arises in computational settings, where the utility function is used in the place of preferences and the modeller needs to approximate the underlying parameter space. The approximation is more costly if false conclusions such as $a \prec_x b$ for some $x \in F$ are to be avoided. This situation does not arise when the space is perfectly normal.

Counterexamples of perfectly normal spaces There are many kinds of parameter space that fail to be perfectly normal. These are useful in delineating the scope of the main theorem of this paper. Some of these, like the lexicographic unit square $[0, 1] \times_{\text{lex}} [0, 1]$, are discussed in subsection 4.4. Others relate to the *product topology* and often arise in the setting where the parameter space consists of beliefs (probability measures) and appear in subsections 4.2 and 4.3.[4] For a simple example where perfect normality

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[4]Recall that for topological space X and Y, the product topology is the smallest topology such that, for any G open in X and H open in Y, the set $G \times H$ is open in $X \times Y$.

fails, consider the product topology on the set $\{0, 1\}^S$ of functions on an arbitrary uncountable set S with values in $\{0, 1\}$. Elements of this set might be interpreted as an uncountable "sequence" of coin tosses.

Following Savage (1972), another interpretation given to a function $f \in \{0, 1\}^S$ is that of an *act*. In a one-person decision problem considered by Savage, Val's preferences are defined on $\{0, 1\}^S$, so that this would correspond to our set A. In a two person setting where the second player also chooses from the same set of acts, a complete, contingent plan for Val would involve defining preferences on A for each of her opponent's strategies $f \in \{0, 1\}^S$. That is, we also have $X = \{0, 1\}^S$. Example 2.2 highlights that, with the product topology on X, Val's preferences may be such that they are not represented by any function that is continuous on X. Depending on the context, there may be ways to fix this problem. For example, one might instead consider the topology generated by the uniform metric $\|x - y\|_\infty = \sup\{|x(s) - y(s)| : s \in S\}$ on X. Note however, that although X is then metrizable (and therefore perfectly normal), it is also discrete (every $x \in X$ is a neighbourhood of itself). This may not be suitable for applications where a more subtle notion of "closeness" is useful.

Perfectly normal spaces of beliefs In some of the applications that appear in section 4, we will let the parameter space be a set of probability distributions on a state space S. For this purpose, we need the following partial extension of Parthasarathy (1967, Theorem 6.4). The relevant definitions appear in remark 2.1 below.

Proposition 2.1. *Let S be a compact set of states. Let Σ be the smallest σ-algebra containing all the zero sets of S, and let $\Delta(S)$ be the set of probability measures on Σ endowed with the weak* topology.*

1. *$\Delta(S)$ is metrizable if and only if S is metrizable.*

2. *If $\Delta(S)$ is perfectly normal, then so is S.*

The proof of part 2 of proposition 2.1 appears in appendix A2. With minor modifications, part 1 follows from Parthasarathy (ibid., Theorem 6.4) and the connection between the Baire σ-algebra and the Borel σ-algebra that we describe below. Missing from the above result is the converse implication: S perfectly normal implies $\Delta(S)$ perfectly normal. Nonetheless, for our present purposes, an important fact that proposition 2.1 reveals is the following:

if S not perfectly normal, then neither is $\Delta(S)$.

When S is compact, Σ of proposition 2.1 is usually referred to as the Baire σ-algebra (the Baire sets). Note that it is more common to let Σ be the smallest σ-algebra that contains every closed set of S (the Borel sets). But recall that when S is perfectly normal, every closed subset is a zero set. Thus, when S is compact and perfectly normal, the Borel sets coincide with the Baire sets.

In contrast, when S is not perfectly normal, there exists some closed subset F of S that is not a zero set. In this case, although every Baire set is a Borel set, the converse does not hold. This is precisely when proposition 2.1 matters, for then the set of beliefs $\Delta(S)$ is not perfectly normal.

Remark 2.1. *When S is finite, Σ is usually the collection of all subsets of S (the discrete topology on S) and $\Delta(S)$ is then identified with the simplex in $\mathbb{R}^S$ with the usual product topology. When the set of states is infinite, it is common to require that the set of states S is measurable and endowed with a σ-algebra Σ of subsets. (Σ can be any collection of subsets that is closed under complementation, countable intersection and countable union.) When considering the set of probability measures $\Delta(S)$ on S, it is common to consider the weak* topology on $\Delta(S)$. The weak* topology is defined as follows. Let $C(S)$ be the set of real-valued, continuous functions on S. Let Σ be a σ-algebra on S. The weak* topology on the set of countably additive probability measures $\Delta(S)$ on (S, Σ) is the smallest topology on $\Delta(S)$ such that, for each $f \in C(S)$, the linear functional $\mu \mapsto \int_S f \, \mathrm{d}\mu$ is continuous on $\Delta(S)$.*

3 Results

In this section we characterise the axioms on preferences of the preceding section through a function that is both a utility at each parameter and continuous across parameters. We also provide a characterisation of a perfectly normal parameter space as a corollary to the main theorem. Connections with results from the literature on jointly continuous representations are made in subsection 3.2. For the case where A is discrete, the representation is shown to be jointly continuous. In subsection 3.3 we build on this result to derive a value function that is continuous in the parameter. This is done via an extension of Berge's theorem of the maximum, which also yields a choice function that is upper hemicontinuous in the parameter.

3.1 Main theorem

The following result is equivalent to the classic representation of a single binary relation by Cantor (1895) in the case that X is a singleton.[5]

Theorem 3.1. *For A countable and X perfectly normal,* (Asy), (NT) *and* (CD) *hold if and only if there exists $U : A \times X \to \mathbb{R}$ such that*

1. *for every x in X, $U(\cdot, x)$ is a utility representation of $\prec_x$ on A; and*

2. *for every a in A, $U(a, \cdot)$ is a continuous function on X.*

The proof of theorem 3.1 appears in appendix A1.

The requirement that X is perfectly normal is essential for the sufficiency of the axioms in theorem 3.1. If X is not a perfectly normal space, there are (possibly many) preferences that satisfy continuous parameter dependence, but have no representation that is continuous in the parameter. The following corollary formalises this statement and its converse.

Corollary 3.1. *X is a perfectly normal if and only if all preferences on a countable set A satisfying* (Asy), (NT) *and* (CD) *have a representation that is continuous in the parameter.*

Proof of corollary 3.1. If X is perfectly normal, then theorem 3.1 completes the sufficiency argument. With minor modifications, the converse follows from the first paragraph of example 2.2.

$\square$

3.2 Joint continuity for discrete alternatives

It is often useful to require that the representation of preferences is jointly continuous on $A \times X$. Examples include consumer demand theory (see subsection 4.1), game theory and Kőszegy and Rabin (2006)'s formalisation of prospect theory. The literature on jointly continuous representations of preferences $\{\prec_x\}_{x \in X}$ includes Kannai (1970), Hildenbrand (1970) and Mas-Colell (1977), Levin (1983) and, more recently, Caterino, Ceppitelli, and Maccarino (2009). For an introductory survey see Mehta (1998). Levin (1983, Theorem 1) provides the simplest and most easily comparable continuity condition for a jointly continuous representation. Instead of (CD), Levin assumes

Axiom JC. *The set $\{(x, a, b) : a \precsim_x b\}$ is closed in $X \times A \times A$.*

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[5]In fact, Cantor's theorem holds for the case where indifference sets are singletons. This latter distinction is minor when $\sim_x$ is an equivalence relation, for Cantor's theorem then applies to the quotient set $A_{/\sim_x}$.

Levin's theorem also requires that X is metrizable and A is a countable union of compact sets. Note that the set $\{(x, a, b) : a \precsim_x b\}$ of (JC) is the graph of $x \mapsto \precsim_x$. This graph is closed if and only if the correspondence $x \mapsto \precsim_x$ is upper hemicontinuous and closed-valued. The latter holds when $\precsim_x$ is closed in $A \times A$ for each $x \in X$.[6] *Upper hemicontinuity* (u.h.c.) holds if, for every closed $F \subseteq A \times A$, the set $\{x : \precsim_x \cap F \neq \varnothing\}$ is closed in X. If we let $F = \{a\} \times B$, for some closed and infinite set B, then (JC) requires that

$$G := \{x : \precsim_x \cap (\{a\} \times B) \neq \varnothing\} = \bigcup_{b \in B} \{x : a \precsim_x b\}$$

is closed in X. Closedness of $\{x : a \precsim_x b\}$ for every $a, b \in A$ is equivalent to (CD). However, (CD) does not imply the above infinite union is closed.

Example 3.1. *Let $X = \mathbb{R}$, $B = \{b_1, b_2, \ldots\}$, and $\{x : a \precsim_x b_n\} = \left[-\frac{n}{n+1}, \frac{n}{n+1}\right]$ for each n. Then the open set $G = (-1, 1)$ is the union of these closed sets. Whilst such preferences are ruled out by (JC), they are compatible with (CD).*

When A is discrete, the following corollary shows that, given (Asy) and (NT), (CD) is necessary and sufficient for a jointly continuous representation.

Corollary 3.2. *If A is discrete, then any function $U : A \times X \to \mathbb{R}$ satisfying condition (2) of theorem 3.1 is jointly continuous.*

The proof of corollary 3.2 appears in appendix A2. This corollary allows us to conclude that, for the case where A is countable and discrete, theorem 3.1 is a generalisation of Levin (1983, Theorem 1) and Caterino, Ceppitelli, and Maccarino (2009, Theorem 4.2). The first reason is straightforward: these theorems require that X is metrizable, whereas we require that it is perfectly normal. The second reason is that corollary 3.2 and example 3.1 imply (JC) is not necessary for a jointly continuous representation.

It remains to compare theorem 3.1 with Caterino, Ceppitelli, and Maccarino (ibid., Theorem 4.1). The latter holds for submetrizable X. X is *submetrizable* if there exists a metric space Y and a continuous bijection $f : X \to Y$. This definition allows the inverse function f^{-1} to be discontinuous. Taking X to be the space $\mathbb{F}$, or the split interval, or the long line (these were introduced in section 2 and are developed in subsection 4.4) yields a perfectly normal parameter space that is not submetrizable (Buzyakova, 2006). This demonstrates the existence of applications to which theorem 3.1 applies, but the other results do not: even when A has just two alternatives.

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[6]When $\precsim_x$ is complete and transitive, this is equivalent to assuming the usual (semi)continuity of $\precsim_x$ at each x: $\{a : a \precsim_x b\}$ and $\{a : b \precsim_x a\}$ are closed in A.

If, on the otherhand, X is not perfectly normal, then corollary 3.1 guarantees the existence of preferences that fail to satisfy condition (2) of theorem 3.1. Since this is necessary for a joint continuity of the representation, the latter also fails to hold for such preferences. Note that this latter argument holds regardless of the cardinality and topology of A.

Finally, corollary 3.2 also demonstrates that in many applications, it is possible to assume (CD) and obtain the same result as (JC). Whilst the requirement that A is discrete and countable is strong, it does allow us to attain the upper bound for the parameter space. Moreover, the results of the next subsection and subsection 4.1 allow us to view A as a salient subset of a larger space A' on which preferences at $x \in X$ are defined. For instance, A might be the set of rational numbers with decimal expansion no greater than 100 places, and A' is the real line. This point is discussed further in subsection 4.1 in a standard consumer choice setting.

3.3   Theorem of the maximum for discrete alternatives

Joint continuity of $U : A \times X \to \mathbb{R}$ is the first premise of Berge's theorem of the maximum. Berge's theorem provides sufficient conditions for U to give rise to a continuous value function $V : X \to \mathbb{R}$. V is the function that selects the supremum of the values $U(\cdot, x)$ takes for each $x \in X$. The same conditions in Berge's theorem also give rise to a u.h.c. (see previous subsection) optimal choice correspondence $\mathcal{C} : X \to 2^A - \varnothing$.

The remaining premises of Berge's theorem relate to the set of constraints on A that Val faces at each $x \in X$. Thus, if $\mathcal{F}(x)$ denotes the set of alternatives available to Val at $x \in X$, then Berge's theorem requires that $\mathcal{F} : X \to 2^A - \varnothing$ is compact-valued, u.h.c., and l.h.c.[7] $\mathcal{F}$ is *compact-valued* if $\mathcal{F}(x)$ is compact for each $x \in X$. When $\mathcal{F}$ is both u.h.c. and l.h.c., it said to be continuous.

In most statements of Berge's theorem, joint continuity of U is simply assumed. In what follows, we will take preferences as primitive. We first consider Berge's theorem of the maximum when the conditions for Levin (1983, Theorem 1) are satisfied.

Theorem 3.2. *For A metrizable and σ-compact, and X metrizable, suppose* (Asy), (NT) *and* (JC) *hold. Then there exists a jointly continuous utility representation $U : A \times X \to \mathbb{R}$ of $\{\prec_x\}_{x \in X}$. Moreover, for any $\mathcal{F} : X \to 2^A - \varnothing$ that is continuous and compact-valued, we have*

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[7]Recall from subsection 2.2 that $\mathcal{F}$ is *lower hemicontinuous* (l.h.c.) if, for every open $G \subset A$, $\{x : \mathcal{F}(x) \cap G \neq \varnothing\}$ is open in X.

1. $V(\cdot) := \max\{U(a, \cdot) : a \in \mathcal{F}(\cdot)\}$ *is a continuous function on* X; *and*

2. $\mathcal{C}(\cdot) := \operatorname{argmax}\{U(a, \cdot) : a \in \mathcal{F}(\cdot)\}$ *is a u.h.c. correspondence on* X.

The proof of theorem 3.2 follows immediately from Levin (1983, Theorem 1) and Ichiishi (1983, Theorem 2.3.1). Given corollary 3.2, it is natural to suppose a similar theorem exists under the conditions of theorem 3.1 provided we assume A is discrete. The only issue is that, when A is discrete, the requirement that $\mathcal{F}$ is continuous is too strong.[8] The following lemma illustrates the problem.

Lemma 3.1. *Let* $A = \{a, b\}$ *and let* $Y = [0, 1]$. *Suppose that* $\mathcal{F} : Y \to 2^A - \varnothing$ *is continuous. Then* $\mathcal{F}$ *is constant.*

The proof of lemma 3.1 appears in appendix A2. When $\mathcal{F}$ is constant, any application to the usual constrained maximisation problems of consumer and producer theory is precluded, for the budget set cannot vary with prices. Fortunately, when A is discrete, we are able to weaken the premises of theorem 3.2 so that $\mathcal{F}$ is u.h.c. instead of continuous. When A is also countable, we may further weaken the premises to get: X perfectly normal instead of metrizable and $\{\prec_x\}_{x \in X}$ satisfying (CD) instead of (JC). The result is the following characterisation.

Theorem 3.3. *For* A *discrete and countable, and* X *perfectly normal,* (Asy), (NT) *and* (CD) *hold if and only if there exists a jointly continuous utility representation* $U : A \times X \to \mathbb{R}$ *of* $\{\prec_x\}_{x \in X}$. *Moreover, for any* $\mathcal{F} : X \to 2^A - \varnothing$ *that is u.h.c. and compact-valued,* (1) *and* (2) *of theorem 3.2 hold.*

The proof of theorem 3.3 appears in appendix A2. The necessity of the axioms in theorem 3.3 is useful for many applied settings where modellers simply posit a jointly continuous utility function. It means that they assume (CD) holds.

4 Applications

In each of the following four subsections, we apply the results of section 3. The first application is to consumer theory where parameters are the product of a standard set of price-wealth vectors with another set of parameters. The secon application is to the literature on "topologies on types" and psychological games. The third application is to belief dependence in a finance setting. The final application is to a setting where preferences exhibit reference dependence and the parameter space is a lexicographically ordered set.

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[8]I thank Maxwell Stinchcombe for bringing this to my attention.

4.1 Consumer theory with a discrete commodity space

A common setting where continuous parameter dependence arises is the theory of consumer demand. Here Berge's theorem of the maximum is used to show that the value function that is indirect utility is continuous and the demand correspondence is u.h.c. (see subsection 3.3) of price-wealth pairs (Sundaram, 1996). This in turn is a stepping stone to envelope theorems and other tools for comparative static analysis.

Let $A \subset \mathbb{R}_+^{n-1}$ be countable and discrete set of commodities. This might be the set $\mathbb{Z}_+^{n-1}$ of vectors with nonnegative integer-valued entries, or with rational entries that have decimal expansions restricted to (at most) 10 decimal places. Whilst this assumption is not classical, it has received relatively recent attention in Milgrom and Segal (2002) and Sah and Zhao (1998) and is standard in the literature on discrete choice in econometrics (McFadden, 1984). It has also been motivated in the game theoretic setting by Aumann and Brandenburger (1995) and Gilboa and Schmeidler (2003).

Let the set of parameters be the cartesian product of the set price-wealth vectors $\mathbb{R}_{++}^n$ with some other set of parameters Θ the observer deems to relevant to Val's decision making. The latter may include other socio-economic data in econometric settings or characterise frames in experiments. Thus $X = \mathbb{R}_{++}^n \times \Theta$, and its elements are denoted by $x = (p, w, \theta)$, where p is the vector of prices $(p_1, \ldots, p_{n-1})$ and w denotes wealth. Val's ability to choose elements of A is constrained by her budget. The budget correspondence varies with the parameter in the following way:

$$\mathcal{B} : X \to 2^A - \varnothing, \quad x \mapsto \{a \in A : p \cdot a \leqslant w\}. \tag{$*$}$$

For each $x \in X$, Val is able to rank the elements of A according to $\prec_x$ with a view to identifying the best element(s) in $\mathcal{B}(x)$. To this end we may suppose $\{\prec_x\}_{x \in X}$ satisfies (Asy) and (NT). This yields a representation $U : A \times X \to \mathbb{R}$ satisfying condition (1) of theorem 3.1. The standard model assumes that Θ is a singleton and that, for all $x, y \in X$, $\prec_x = \prec_y$. The most natural generalisation would let preferences vary across Θ.

Since A is discrete, $\mathcal{B}(x)$ is compact if and only if it is finite for each x. When this holds, (Asy) and (NT) are sufficient for a maximal element to exist. If Val is indifferent between two or more best elements, these will all lie in her demand correspondence at x. The latter is a map $\mathcal{D} : X \to 2^A - \varnothing$ such that $\mathcal{D}(x) \subseteq \mathcal{B}(x)$ for all $x \in X$.

For the purposes of conducting a comparative statics analysis a minimal requirement is that there exists a continuous indirect utility function $V : X \to \mathbb{R}$, $x \mapsto \max\{U(a, x) : a \in \mathcal{B}(x)\}$, and that $\mathcal{D}$ is u.h.c. (For instance, the latter ensures that the demand correspondence is continuous whenever

it is a function.) For these properties to hold, theorem 3.3 requires that in addition to assuming (CD), we must show that X is perfectly normal, and $\mathcal{B} : X \to 2^A - \varnothing$ is compact-valued and u.h.c.

Provided we assume Θ is perfectly normal, the set $X = \mathbb{R}_{++}^n \times \Theta$ is perfectly normal by (Tkachuk, 2011, p.249). Indeed, this is true of any cartesian product of a second countable space and a perfectly normal space.[9] The application of the present model of preferences to the consumer demand setting is therefore complete provided we can show the budget correspondence $\mathcal{B}$ is sufficiently well-behaved. This task is a straightforward consequence of the following two assumptions: A is a discrete subset of $\mathbb{R}_+^{n-1}$ and, by $(*)$, $\mathcal{B}$ is independent of $\theta \in \Theta$.[10]

4.2   Topologies on types

The present results are important for the literature on topologies on types (Chen et al., 2010; Dekel, Fudenberg, and Morris, 2006; Mertens and Zamir, 1985). For instance, in Mertens and Zamir (1985) the main assumption regarding the space S is compactness. The authors interpret a point in S as a "full listing of the stategy spaces and payoff functions [of the players in a game]". They then define a hierarchy of beliefs to be a sequence $\Delta_0, \Delta_1, \Delta_2, \ldots$ such that Δ_0 is a compact subset of S, and for each $k > 0$, Δ_k is a compact subset of $\Delta_{k-1} \times [\Delta(\Delta_{k-1})]^n$. Since S need not be perfectly normal, and moreover, the product of perfectly normal spaces is not, in general, perfectly normal, proposition 2.1 in conjunction with corollary 3.1 tell us that the utility functions allowed by the model may not reflect the underlying parametric continuity of preference. Chen et al. (2010) and Dekel, Fudenberg, and Morris (2006) consider the special case where S is finite and there are two players. The metrics they identify on the space of types ensure the resulting type space is perfectly normal.

The literature on psychological games constructs a similar hierarchy of beliefs. Our alternatives correspond to their outcomes, and our parameters

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[9]The topology of a second countable space is generated by a countable collection of basic open sets. The argument that $\mathbb{R}_{++}^n$ is second countable, is a straightforward extension of the fact that the set of open intervals with rational endpoints generate the usual open sets of $\mathbb{R}$.

[10]The proof is as follows. Let D be a directed set and let $(x^\nu, a^\nu)_{\nu \in E}$ be any net such that $(x^\nu, a^\nu) \in \{x^\nu\} \times \mathcal{B}(x^\nu)$ for all $\nu \in E$ and $\lim(x^\nu, a^\nu) = (x, a) \in X \times A$. Then, since A is discrete, there exists ν' such that for all $\nu \geqslant \nu'$, $a^\nu = a$. Let $x^\nu = (p^\nu, w^\nu, \theta^\nu)$ for each $\nu \in E$. Then, by $(*)$, for all $\nu \geqslant \nu'$, we have $p^\nu \cdot a^\nu = p^\nu \cdot a$ and $p^\nu \cdot a \leqslant w^\nu$, so that indeed $p \cdot a \leqslant w$ as a consequence of continuity of $\leqslant$ on $\mathbb{R}$. This ensures that the graph of $\mathcal{B}$ is closed a condition which is sufficient for u.h.c. (see the proof of theorem 3.3 for more on this latter connection).

correspond to their beliefs. (On Geanakoplos, Pearce, and Stacchetti (1989, p.65) it is stated that "payoffs for player i are defined first on the outcomes (given any belief profile b) and only afterward extended ...".) Similar to Chen et al. (2010) and Dekel, Fudenberg, and Morris (2006), Geanakoplos, Pearce, and Stacchetti (1989) assumes the basic set S on which beliefs are constructed is finite. They are therefore able to obtain a metrizable hierarchy of beliefs. In a related paper, beginning with preferences Gilboa and Schmeidler (2003) provide a decision theoretic version with no explicit hierarchy of beliefs. Their axioms are a superset of ours. Their assumptions also ensure the set of beliefs is metrizable.

4.3 Continuous belief dependence and portfolio choice

Consider a setting where Val's alternatives $a \in A$ consist of investment portfolios. Consider an investment advisor wishes to check whether Val's choice of portfolio is robust to perturbations of Val's beliefs. Once Val has ranked the portfolios or chosen one, the investment advisor presents charts where each one plots the evolution of a path representing a portfolio's values across time. When is this form of perturbation closely related to perturbing beliefs? Moreover, suppose the investment manager has elicited a utility representation of Val's preferences $\{\prec_x\}_{x \in X}$. In settings where (CD) appears to be reasonable, when is the representation a continuous function of beliefs? A large class of examples where the answers to both these questions is yes is provided by part 1 of proposition 2.1.

The set $X = \Delta(S)$ denotes the set of beliefs that Val might have. Val's preferences $\{\prec_x\}_{x \in X}$ over a set of portfolios A are indexed by her beliefs. If S is metrizable, then $\Delta(S)$ is metrizable and hence perfectly normal. Then theorem 3.1 applies whenever axioms (Asy), (NT) and (CD) hold. This yields a function $U : A \times X \to \mathbb{R}$ that is a utility function on A for each belief, and is continuous in beliefs. Moreover, as the topology on S is embedded in that of X, perturbations in the latter are indeed close to those of the former. In this case, the actions of investment managers are justified.

On the other hand, when S is not perfectly normal, the contrapositive of proposition 2.1 tells us that $\Delta(S)$ is not, and corollary 3.1 then provides a warning to the investment advisor who has elicited a representation $U : A \times X \to \mathbb{R}$ of preferences. He should be aware that a discontinuous $U(a, \cdot)$, for some a, does not imply discontinuous dependence of Val's preferences. Or, in the case that alternatives are discrete (as in section 4.1), the indirect utility function of beliefs may reveal discontinuities that are absent in Val's preferences.

Two constructions of Wiener measure   We now provide a more detailed demonstration of the relevance of proposition 2.1 and the results of section 3 for the financial setting. We draw on Nelson (1965) and Taylor (2006) who construct the familiar (Wiener) probability measure of a Brownian motion in two related, yet distinct, ways. The first yields a compact and metrizable state space and set of beliefs, whereas the second yields a state space (and hence set of beliefs) that is not perfectly normal.

Let $\mathbb{Q}_+$ denote the nonnegative rational numbers and let $\dot{\mathbb{R}}^n$ denote the one-point compactification of $\mathbb{R}^n$, that is $\dot{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Taylor (2006, Ch.16) defines Wiener measure on the set of paths

$$\mathfrak{B} = \prod_{t \in \mathbb{Q}_+} \dot{\mathbb{R}}^n.$$

As a product of compact spaces, $\mathfrak{B}$ is compact. Moreover, since $\dot{\mathbb{R}}^n$ is metrizable, so is the countable product $\mathfrak{B}$. Proposition 2.1 then ensures that the set of probability measures $\Delta(\mathfrak{B})$ is metrizable. This means that we may let the set of Val's beliefs to be $X = \Delta(\mathfrak{B})$ and define preferences $\{\prec_x\}_{x \in X}$ on a set of portfolios A, and provided the axioms apply, theorem 3.1 ensures a continuous representation exists.

One special belief that Val might have is Wiener measure. Consider the set $\mathfrak{B}_0$ of uniformly continuous paths in $\mathfrak{B}$. The set $\mathfrak{B}_0$ is important, for its elements have unique extension to the set $C(\mathbb{R}_+; \mathbb{R}^n)$ of continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^n$ and since the path of a Brownian motion is almost surely continuous, Wiener measure must assign probability one to $\mathfrak{B}_0$. A fortiori, Wiener measure must therefore be defined on $\mathfrak{B}_0$. Taylor (ibid., Proposition 16.3) shows that $\mathfrak{B}_0$ is indeed a Borel set. Since $\mathfrak{B}$ is metrizable, it is perfectly normal, and therefore $\mathfrak{B}_0$ is also in the Baire σ-algebra.

A different approach to the construction of Wiener measure is taken by Nelson (1965). Instead of starting with paths on the rationals and extending to the reals as above, we consider the set of paths $\widetilde{\mathfrak{B}}$ consisting of all functions on $\mathbb{R}_+$ with values in $\dot{\mathbb{R}}^n$. The relevant topology is, as before, the product topology. As a product of compact sets, $\widetilde{\mathfrak{B}}$ is compact. However, in this case, although the set $C(\mathbb{R}_+; \mathbb{R}^n)$ is a Borel set, it is not in the Baire σ-algebra on $\widetilde{\mathfrak{B}}$. As a result, the latter is not perfectly normal, and, although it is still possible to define Wiener measure, proposition 2.1 implies that the set of probability distributions $\Delta(\widetilde{\mathfrak{B}})$ will not be perfectly normal. So if $Y = \Delta(\widetilde{\mathfrak{B}})$, corollary 3.1 implies that there exist preferences $\{\prec_y\}_{y \in Y}$ satisfying (Asy), (NT) and (CD) for which any function $U : A \times Y \to \mathbb{R}$ that is a utility function on A, for each $x \in X$, is discontinuous on X.

4.4 Lexicographic reference dependence

This subsection develops a detailed example showing that the results apply to problems that cannot be modelled using either a parameter-free utility function or the pre-existing results in the literature on jointly continuous utility representations. To fix ideas, we develop the example within a framework of reference-dependent preferences, as in Kőszegy and Rabin (2006). The distinctive feature of our model is that the parameter space is the lexicographically ordered set $[0,1] \times_{\text{lex}} \{0,1\}$ that was introduced in section 2 as the split interval. We then extend the example to provide another where there is no continuous representation even though preferences satisfy (CD).

Our choice of parameter space is important, for it is a leading example of a compact space that is perfectly normal, but not metrizable. Since it is also homeomorphic to the space $\mathbb{F}$ of *increasing* functions on $[0,1]$ with values in $\{0,1\}$ with the topology of pointwise convergence (also introduced in section 2), the results also apply to the setting where $\mathbb{F}$ is the set of parameters. The same applies to the long line that is formally defined at the and of the subsection.

Reference dependent preferences Let $A = \{a,b\}$ be the set of prospects (alternatives), and let $X = [0,1] \times \{0,1\}$ be the set of reference points (parameters). Assume that (Asy) holds, and that $\{x : a \sim_x b\} = I \times \{0,1\}$ for some closed and nondegenerate interval $I \subseteq [0,1]$. As in example 2.1, (NT) trivially holds since $|A| = 2$.

Lexicographic ordering of reference points Consider the lexicographic ordering $<_{\text{lex}}$ over X that ranks $x \in X$ higher than $y \in X$ if and only $y_1 < x_1$ or $[y_1 = x_1$ and $y_2 < x_2]$. The ordering $<_{\text{lex}}$ need not reflect Val's "preferences" over reference points in general, and in any case, it is distinct from her preferences $\{<_x\}_{x \in X}$ on A.

Order topology on reference points For any $y, z \in X$, each of $\{x : y <_{\text{lex}} x\}$ and $\{x : x <_{\text{lex}} z\}$ is an open order interval of X. From these basic sets, we derive a topology τ by taking unions of finite intersections of such intervals generates the lexicographic order topology on X. Because of the discrete nature of the second dimension of X, an arbitrarily small open neighbourhood of a point $y = (y_1, 1)$ is of the form $\{x : y \leqslant_{\text{lex}} x <_{\text{lex}} z\}$, for some z satisfying $y_1 < z_1$. This neighbourhood is of the form

$$\Big((y_1, z_1) \times \{0,1\} \Big) \cup \{y\} \tag{$**$}$$

Similarly, when $y_2 = 0$, small enough perturbations consider $x \leqslant_{\mathrm{lex}} y$, so that $x_1 < y_1$. (See Vaughan (1971) for a more detailed discussion.)

Continuous reference dependence The collection of sets τ is distinct from the one obtained by considering X as a subspace of $\mathbb{R}^2$ with the usual, Euclidean topology. It is also distinct from the topology obtained by considering X as a subspace of the $\mathbb{R}^2_+$ with the lexicographic order topology (Munkres, 2000, p.107)). Nonetheless, it may be reasonable to suppose that Val's preferences on A satisfy (CD) with respect to τ. (CD) holds if the sets $\{x : a \prec_x b\}$ and $\{x : b \prec_x a\}$ are members of τ. The following lemma confirms this is the case.

Lemma 4.1. *The set $\{x : a \sim_x b\}$ is closed in X.*

Proof of lemma 4.1. This follows because $\{x : a \sim_x b\} = I \times \{0,1\}$ and, by Hernández-Gutiérrez (2013, Proposition 2.1), the latter is homeomorphic to $X = [0,1] \times \{0,1\}$ for every closed and nondegenerate $I \subseteq [0,1]$. $\qquad\square$

Representation of preferences With the topology τ, the set X is a well known example of a perfectly normal topological space that is not metrizable (Gruenhage and Moore, 2011). Thus, by theorem 3.1, there is a representation of Val's preferences that characterises continuous parameter dependence. By continuous parameter dependence, in this setting, we mean that for every $x \in X$ such that $a \prec_x b$, there exists an open order interval of $\prec_{\mathrm{lex}}$ of the form $(**)$ containing x such that $a \prec_y b$ for every y in that interval. That is, a function $U : A \times X \to \mathbb{R}$ such that, for each $x \in X$, $U(\cdot, x)$ is a utility function on A and, for each $a \in A$, $U(a, \cdot)$ is continuous on X.

Other approaches do not apply Since X is not metrizable, it does not satisfy the conditions Levin (1983). (Similarly, Caterino, Ceppitelli, and Maccarino (2009) does not apply.) Thus, the pre-existing results on joint continuity do not apply. Futhermore, the fact that $(x_1, 0) \prec_{\mathrm{lex}} (x_1, 1)$ for each $x_1 \in [0,1]$ means that there is no real-valued utility representation of $\prec_{\mathrm{lex}}$. (Any representation must have uncountably many open gaps (Debreu, 1964).) In turn, this implies that there is no utility representation of any preference that Val might have over the yet larger set $A \times X$.

Generalising the example By theorem 3.1, the above example can immediately be extended to a countable number of alternatives A. Somewhat surprisingly, the next proposition confirms that the present example cannot be extended so that the second dimension contains any other elements

$0 < x_2 < 1$. Any such parameter space (including the full unit square $[0,1]^2$) fails to be perfectly normal, so that, by corollary 3.1, continuous reference dependence may not be characterised by a continuous function.

Proposition 4.1. *Let* $Y = [0,1] \times \left\{0, \frac{1}{2}, 1\right\}$ *be endowed with the order topology generated by* $<_{\text{lex}}$*. There is no continuous function* $f : X \to \mathbb{R}$ *such that* $f^{-1}(0) = I \times \{0,1\}$*.*

The proof of proposition 4.1 appears in appendix A2. Hernández-Gutiérrez (2013, Proposition 2.1) strengthens this result further, for any closed subset of $[0,1] \times \{0,1\}$ that has no isolated points is homeomorphic to $I \times \{0,1\}$ and as such there is a large class of parameter dependent preferences that do not have a continuous representation.

The long line   We conclude the section with another example of a perfectly normal, but nonmetrizable set. It too involves a lexicographic ordering over the set of parameters.

Example 4.1. *Let* X' *be the cartesian product of the set* $\mathbb{Z}_+$ *of nonnegative integers with the half open interval of real numbers* $[0,1)$*. Then* $X' = \mathbb{Z}_+ \times [0,1)$ *is homeomorphic to* $\mathbb{R}_+$ *provided it is endowed with the topology generated by the intervals* $\{x : x <_{\text{lex}} y\}$ *and* $\{x : z <_{\text{lex}} x\}$ *of* $<_{\text{lex}}$*. Since* $\mathbb{R}_+$ *is metrizable, so is* X'*.*

Recall that $\mathbb{Z}_+$ *is the smallest well-ordered infinite set. Its supremum is the first infinite ordinal number, and is often denoted by* ω*, and so we may write* $\mathbb{Z}_+ \equiv [0, \omega)$*. An extension of this is the well-ordered set* $[0, \omega_1)$ *of all countable ordinal numbers. The supremum* ω_1 *is the first uncountable ordinal. With the above lexicographic order* $<_{\text{lex}}$ *and corresponding topology, the product* $X'' = [0, \omega_1) \times [0,1)$ *yields a set that is perfectly normal, but not metrizable and is known as the "long line".*

In contrast, with $\mathbb{R}_+ \cup \{\omega\}$ *however, the one-point compactification* $X'' \cup \{\omega_1\}$ *is not perfectly normal.*

5   Summary

We have given conditions on preferences and the parameter space for a general model of parametric continuity of preference. The main theorem shows that preferences satisfying the axioms can be represented by a function that is a utility given the parameter and is continuous on the parameter space.

Whilst the main drawback of the present model is that the set of alternatives must be countable, this assumption has allowed us to obtain the

minimal conditions for parametric continuity. Firstly, the axioms on preferences are necessary and sufficient for parametric continuity of the representation. Secondly, if the parameter space does not satisfy the conditions for the main theorem, then there exist preferences that vary continuously with the parameter, but have representation that is continuous in the parameter.

When the set of alternatives has the discrete topology, (CD), the axiom that captures continuous parameter dependence, is both necessary and sufficient for joint continuity of the representation on the product of alternatives and parameters. This yields a generalisation of the existing results from the literature on jointly continuous representations. Via a simple extension of Berge's theorem of the maximum, this joint continuity allowed us to derive (i) a value function that is continuous and (ii) a choice correspondence that is upper hemicontinuous. If the parameter space is not perfectly normal, there are always preferences that satisfy (CD) and violate (i).

The first application considered a setting of consumer choice, where the parameter space is the product of price-wealth vectors with an abstract space of frames, types or other parameters. For the case where commodities are discrete, the joint continuity of the representation allows us to obtain continuity of the indirect utility function and upper hemicontinuity of the demand correspondence. That is to say, provided the set of parameters that we use to extend the standard model is perfectly normal, the indirect utilty function and demand correspondence are well-behaved.

Our second application was to the "topologies on types" literature and psychological games. We saw that the Mertens and Zamir (1985) model of types, which only requires compactness of the state space allows for preferences with utility representations that depend discontinuously on the parameter, even (CD) is satisfied. The other models all ensured the state space or basic parameter space is finite, so that the set of types or beliefs is metrizable. Our results confirm that in these cases, perturbations of types are well-defined for utility representations of preferences.

Our third application considered portfolio choice with belief dependence of preferences. When the set of states is metrizable, then so is the set of beliefs. This ensures that perturbations of states are topologically similar to perturbations of beliefs. Moreover, theorem 3.1 applies, and this ensures that perturbing the utility function with respect to beliefs or states is equivalent to perturbing preferences. By contrast, the contrapositive of proposition 2.1 in conjunction with corollary 3.1 shows that there exist state spaces that give rise to continuous belief dependence that cannot be characterised with a continuous function. We argue that these results are important for investment managers who may not be able to observe beliefs, but wish to check a choice of portfolio is robust. These results are relevant even in the

standard setting one encounters in mathematical finance, where prices are driven by a Brownian motion.

Our final application considered reference dependence, where the parameter space is the set of reference points. The topology on reference points that we considered was generated by a lexicographic ordering. This example serves to distinguish the present results from other models including parameter-free approaches and the existing results on jointly continuous representations. Extending the same example to a larger set of parameters yielded a large class of preferences for which continuous parameter dependence is not characterisable. From an applied perspective, this example serves to highlight the following: (i) the relevant topology need not be the usual Euclidean one; and (ii) even simple topologies may not be metrizable. These observations are important in checking robustness of strict preference and for capturing this property with a utility function that is continuous in the parameters.

A1 Proof of theorem 3.1

The necessity of axioms (Asy) and (NT) for part (1) of theorem 3.1, is implied by classical (parameter-free) representation theorems for each x. Moreover, lemma 2.1 confirms that (CD) is also necessary.

Sufficiency of the axioms Let $\{1, 2, 3 \dots\}$ be an arbitrary enumeration of A, and by $[j]$ we will denote the subset of A that contains the first j elements of the enumeration. By $U^j : [j] \times X \to \mathbb{R}$ we will denote the utility representation of the projection of preferences $\{\prec_x : x \in X\}$ onto the first j elements of the enumeration. That is, if we recall that for each $x \in X$, $\prec_x$ is a subset of $A \times A$, then we see that $\{\prec_x : x \in X\} \subset (A \times A)^X$. Hence by the projection of preferences onto $[j]$ we mean $\{\prec_x : x \in X\} \cap ([j] \times [j])^X$ which is a well defined intersection since

$$([j] \times [j])^X \cap (A \times A)^X = \prod_{x \in X} ([j] \times [j]) \cap (A \times A)$$
$$= \prod_{x \in X} ([j] \cap A) \times ([j] \cap A)$$
$$= ([j] \times [j])^X.$$

We use this projection to proceed by induction on A. For the basic case, let $U^1(1, x) = 0$ for all $x \in X$. By condition (Asy), U^1 is a representation for the projection of preferences onto $[1] \times [1]$ and it is clearly continuous. This completes the proof for the basic case. The induction hypothesis is

the following. Suppose that for some $j \geqslant 1$, there exists a representation U^{j-1} of the projection of preferences onto $[j-1]$. From this we obtain a representation of the projection onto $[j]$.

For $a \in [j-1]$ let $U^j(a, \cdot) = U^{j-1}(a, \cdot)$. By the induction hypothesis, for all $a, b \in [j-1]$ and $x \in X$ we have,

$$a \prec_x b \quad \Leftrightarrow \quad U^j(a, x) < U^j(b, x),$$

and on $[j-1]$, U^j is continuous. To complete the inductive step, we must select a continuous function $U^j(j, \cdot)$ on X such that for each x, $U^j(\cdot, x) : [j] \to \mathbb{R}$ represents $\prec_x \cap ([j] \times [j])$.

Summary of inductive step   Define upper and lower envelopes, g and h respectively, of $U^j([j-1], X)$ relative to alternative j. Check this pair of functions satisfy the conditions for Michael's selection theorem (the version by Good and Stares (2000) that was defined in section 2). First, $g : X \to \mathbb{R}$ is weakly dominated by $h : X \to \mathbb{R}$ pointwise; second, they are equal if and only if for some k in $[j-1]$, $j \sim_x k$; third they are respectively upper and lower semi-continuous. This, together with the fact that X is perfectly normal, implies, via Michael's selection theorem, that the required function $U^j(j, \cdot)$ exists.[11]

Definition of upper and lower envelopes   The following step is commonly taken in the construction of envelopes. We introduce two fictional alternatives $\underline{a}$ and $\overline{a}$, such that for all $x \in X$ and $k \in [j]$, we have $\underline{a} \prec_x k \prec_x \overline{a}$. Accordingly, we define $[j-1]^* := [j-1] \cup \{\underline{a}, \overline{a}\}$, and for each $x \in X$, let $U^j(\underline{a}, x) = -\infty$ and $U^j(\overline{a}, x) = +\infty$. Both are clearly continuous functions from X into the extended real line.[12] Moreover, for all $x \in X$, there exists $k, l \in [j-1]^*$ such that $k \lesssim_x j$ and $j \lesssim_x l$, so the following are well defined:

$$g(x) := \max \left\{ U^j(k, x) : k \lesssim_x j \text{ and } k \in [j-1]^* \right\},$$
$$h(x) := \min \left\{ U^j(k, x) : j \lesssim_x k \text{ and } k \in [j-1]^* \right\}.$$

Applying Michael's selection theorem   The following three lemmata ensure that g and h satisfy the conditions of g and h respectively in Michael's selection theorem.

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[11]Recall, we choose continuous $f : X \to \mathbb{R}$ such that $g \leqslant f \leqslant h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$, where g and h are respectively upper and lower semi-continuous.

[12]The fact that they do not take values in $\mathbb{R}$ is of no concern for the final representation as Michael's selection theorem ensures that we only select a function with some value equal to either of these if $U^j(\underline{a}, x) \leqslant U^j(\overline{a}, x)$ holds with equality, which is clearly impossible.

Lemma A1.1. *For all $x \in X$, $g(x) \leqslant h(x)$.*

Proof of lemma A1.1. On the contrary, suppose that for some $x \in X$, $h(x) < g(x)$. Then, by construction, there exists $k, l \in [j - 1]$ such that $k \precsim_x j$, $j \precsim_x l$. But since

$$h(x) := U^j(l, x) < U^j(k, x) =: g(x),$$

we either have a violation of (NT), or a violation of the induction hypothesis (that $U^{j-1}(\cdot, x)$ was order-preserving at each x on $[j - 1]$). □

Lemma A1.2. *For all $x \in X$: $g(x) = h(x)$ iff for some $k \in [j - 1]$, $k \sim_x j$.*

Proof of lemma A1.2. If $g(x) = h(x)$, then, by construction, there is some $k \in \{l : l \precsim_x j\} \cap \{l : j \precsim_x l\}$, and by (Asy), for every l in the intersection of these sets $l \sim_x j$. Conversely, if $j \sim_x k$, then both $k \precsim_x j$ and $j \precsim_x k$. □

Lemma A1.3. *$g : X \to \mathbb{R}$ is an upper semicontinuous function.*

A symmetric argument to the one that follows, but with inequalities and direction of weak preference reversed, shows that h is lower semicontinuous.

Proof of lemma A1.3. Recall (or see Kelley (1975, p.101)) that g is upper semicontinuous provided the set $\{x : r \leqslant g(x)\}$ is closed for each $r \in \mathbb{R}$. Note that by construction of g and the definition of maximum,

$$\{x : r \leqslant g(x)\} = \bigcup_{k \in [j-1]^*} \left(\{x : r \leqslant U^j(k, x)\} \cap \{x : k \precsim_x j\}\right)$$

which is a finite union of closed sets by the following arguments: firstly $U^j(k, \cdot)$ is continuous, so that $\{x : r \leqslant U^j(k, x)\}$ is closed (preimage of a closed set is closed); and secondly, $\{x : k \precsim_x j\}$ is closed by (CD). Since the finite union of closed sets is closed the proof of claim A1.3 is complete. □

The countably infinite case The above argument holds for each j in $\mathbb{N}$, and since we have been choosing functions using Michael's selection theorem for each j, we may extend to all of $\mathbb{N}$ by appealing to the axiom of dependent choice. This ensures that we may define $U : A \times X \to \mathbb{R}$ as follows

$$U = \bigcup_{j \in \mathbb{N}} U^j.$$

This extension to the countably infinite case is equivalent to the argument of Kreps (1988, p.23). There, we instead extend to the countably infinite case using a diagonal argument $U(j, \cdot) = U^j(j, \cdot)$ for each $j \in \mathbb{N}$, and appeal to the axiom of (dependent) choice.[13]

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[13]I thank Atsushi Kajii for bringing this subtle issue to my attention.

A2 Remaining proofs

Proof of proposition 2.1. For each $s \in S$, the Dirac measure δ_s on Σ is the function that assigns value 1 to any set that contains s and is zero otherwise. The preimage of the open set $\{r \in \mathbb{R} : r \neq 1\}$ under δ_s is equal to the union of all sets that do not contain s; it is therefore equal to $S - \{s\}$, and is therefore open. Thus, the mapping $s \mapsto \delta_s$ is thereby continuous and injective. Recall that every continuous injection that is a closed map is also an embedding. That is, a homeomorphism onto its image. Thus, if the image $\delta_S \subset X$ of S under δ is closed, and we will have shown that there exists a subspace of X that is not perfectly normal. This implies that X itself fails to be perfectly normal, for every subspace of a perfectly normal space inherits the same property.

The following argument shows that δ_S is indeed closed. Consider any net $\{\delta_s\}$ in δ_S converging weakly to μ in $\mathcal{P}(S)$. By the definition of weak convergence, $\int_S f d\delta_s \to \int_S f \, \mathrm{d}\mu$, for each continuous $f : S \to \mathbb{R}$. Now $\int_S f \, \mathrm{d}\delta_s = (f_*\delta_s)(\mathbb{R}) = f(s)$ for each s, and since f is continuous and S is compact, $f(s)$ converges to some k in the image of f. Hence, $\int_S f d\mu = k$ and μ lies in $\delta(S)$ and the proof is complete. $\qquad\square$

Proof of corollary 3.2. Fix $(a, x) \in A \times X$ and consider, for some directed set D, a net $E = ((a_\nu, x_\nu))_{\nu \in D}$ in $A \times X$ with limit (a, x). We show that $U(a_\nu, x_\nu) \to U(a, x)$. Recall that (a, x) is the limit of E if and only if, for every neighborhood N of (a, x), there exists $\mu \in D$ such that for every $\nu \geqslant \mu$, $(a_\nu, x_\nu) \in N$. Since A is discrete, $\{a\}$ is open and for some N_x open in X, the set $\{a\} \times N_x$ is an (open) neighborhood of (a, x) in the product topology on $A \times X$. Thus, there exists μ such that for every $\nu \geqslant \mu$, $U(a_\nu, x_\nu) = U(a, x_\nu)$. Finally, condition 2 of theorem 3.1 ensures that $U(a, x_\nu) \to U(a, x)$. $\qquad\square$

Proof of lemma 3.1. Suppose otherwise. In particular suppose that for some $x \in [0, 1]$, $\mathcal{F}(x) = B$ and for some $y > x$, $\mathcal{F}(y) \neq B$.

The first case is $B = A$. Then since $\mathcal{F}(y) \neq \varnothing$, without loss of generality, suppose $\mathcal{F}(y) = \{a\}$ and let $G := \{a\}$. Let $F^+(G)$ denote $\{z : \mathcal{F}(z) \subset G\}$ and let $F^-(G)$ denote $\{x : \mathcal{F}(x) \cap G \neq \varnothing\}$. If $F^+(G)$ is open, then $\mathcal{F}$ is u.h.c. However, by Ichiishi (1983, p.32), $F^+(G) = X - F^-(A - G)$, so that $F^-(A - G) = F^-(\{b\})$ is closed. It is also nonempty, for $x \in F^-(\{b\})$ and not equal to $[0, 1]$ as it does not contain y. But then it fails to be open, a contradiction of the assumption that $\mathcal{F}$ is l.h.c. A similar argument shows that u.h.c. fails whenever l.h.c. holds.

The second case is where $B \neq A$ for all $z \in [0, 1]$. In this case, the same argument shows that u.h.c. and l.h.c. cannot simultaneously hold. $\qquad\square$

Proof of theorem 3.3. First note that, since A is discrete, $\mathcal{F}(x)$ is compact iff $\mathcal{F}(x)$ is finite. For (1), the proof of Ichiishi (1983) requires upper semicontinuity of $(a, x) \mapsto U(a, x)$ and u.h.c. of $\mathcal{F}(\cdot)$. By corollary 3.2, U is jointly continuous, and so the proof of (1) follows from that of (2).

Since $\mathcal{C} = \mathcal{C} \cap \mathcal{F}$, and $\mathcal{F}$ is u.s.c. and compact valued, (2) follows from lemma 2.2.2 and theorem 2.2.3 of Ichiishi (ibid.), provided that the graph of $\mathcal{C}$ is closed in $X \times A$. The graph of $\mathcal{C}$ is the set $\operatorname{gr}\mathcal{C} = \{(x, a) \in X \times A : a \in \mathcal{C}(x)\}$. For any directed set D, let $(x_\nu, a_\nu)_{\nu \in D}$ be any net with values in $\operatorname{gr}\mathcal{C}$ and limit equal to $(\bar{x}, \bar{a})$. Since A is discrete, the singleton set $\{a\}$ is the smallest open neighbourhood of any $a \in A$. Thus $(x_\nu, a_\nu)_{\nu \in D}$ satisfies the property that for some $\mu \in N$, $a_\nu = \bar{a}$ for all $\nu \geqslant \mu$. Thus, (x_ν, a_ν) is eventually in $G \times \{\bar{a}\}$ for some open $G \subset X$. Since $\mathcal{C}(x_\nu) \subseteq \mathcal{F}(x_\nu)$ for every $\nu \in D$, $\bar{a} \in \mathcal{F}(x_\nu)$ for every $\nu \geqslant \mu$. Since $\mathcal{F}$ is u.s.c., its graph is closed, and $(\bar{x}, \bar{a}) \in \operatorname{gr}\mathcal{F}$. Then by joint continuity of U, $\lim_\nu U(x_\nu, a_\nu) = U(\bar{x}, \bar{a})$. Now suppose that there exists $a \in A$ such that $U(\bar{x}, \bar{a}) < U(\bar{x}, a)$, so that $\bar{a}$ is not an argmax at $\bar{x}$. But this would contradict the assumption that $\bar{a}$ is an argmax for all $\nu \geqslant \mu$. Thus $(\bar{x}, \bar{a}) \in \operatorname{gr}C$, as required. □

Proof of proposition 4.1. Recall that an arbitary continuous function $f : X \to \mathbb{R}$ satisfies the property that $G_n = \left\{x : |f(x)| < \frac{1}{n}\right\}$ is open for each $n \in Z_{++}$. We show that $\bigcap_1^\infty G_n \neq F$. The proof is immediate unless $F \subseteq G_n$ for each $n \in \mathbb{Z}_{++}$, so suppose this case holds. Then, by Hernández-Gutiérrez (2013, Proposition 2.1) F is homeomorphic to the compact set $[0, 1] \times \{0, 1\}$. Thus, for each n, G_n can be taken to be the union of finitely many open sets. Every element y that lies between the upper and lower bound of F, has a neighbourhood of the form of equation $(\ast\ast)$. Thus, each G_n contains all but finitely many elements of the set $I \times \{\frac{1}{2}\}$. Since there are only countably many $n \in \mathbb{Z}_{++}$, the deletion of a countable union of a finite number of points is still countable, so the intersection $\bigcap_1^\infty G_n$ contains elements of $I \times \{\frac{1}{2}\}$. □

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