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**TITLE: A Two-Parameter Model of  
Dispersion Aversion**

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## A two-parameter model of dispersion aversion.

### Abstract

The idea of representing choice under uncertainty as a trade-off between mean returns and some measure of risk or uncertainty is fundamental to the analysis of investment decisions. In this paper, we show that preferences can be characterized in this way, even in the absence of objective probabilities. We develop a model of uncertainty averse preferences that is based on a mean and a measure of the dispersion of the state-wise utility of an act. The dispersion measure exhibits positive linear homogeneity, sub-additivity, translation invariance and complementary symmetry. Since preferences are only weakly separable in terms of these two summary statistics, the uncertainty premium need not be constant. We show that the standard results originally derived in the context of mean-variance analysis and expected utility theory apply in this more generally setting. In particular, we generalize the concept of decreasing absolute risk aversion and show that the usual comparative static results from EU theory remain valid. Further we derive two-fund separation and asset pricing results analogous to those that hold for the standard CAPM.

**Keywords:** uncertainty aversion, mean utility, dispersion of utility, weak-separability

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# 1 Introduction: dispersion versus return

Ever since the pioneering work of Markowitz (1952) and Tobin (1958), the idea of representing investment decisions in terms of a trade-off between risk (often characterized by some measure of the dispersion or variation of the return) and expected return has played a dominant role in finance theory. The mean-variance analysis presented by Markowitz and Tobin formed the basis of the Capital Asset Pricing Model (CAPM) (Sharpe 1964, Lintner 1965) which remains the main workhorse of financial analysis. However, mean-variance analysis has been subject to a wide range of criticisms. The first criticism to arise came from proponents of expected utility theory (EUT), who observed that mean-variance analysis was consistent with EUT only for the special (and unattractive) case of a quadratic utility function.<sup>1</sup> If the EUT hypothesis is abandoned, it is possible to consider more general mean-variance preferences, but these are typically ad hoc functional forms, lacking the axiomatic foundations that characterize EUT.

A more recent set of criticisms relates to the choice of the variance as the measure of risk. While the variance has appealing qualities, these are most evident for the case of normal distributions, which are fully characterized by the mean and standard deviation. A large body of evidence suggests that the return distributions for many assets are ‘fat-tailed’ having excess kurtosis relative to the normal. This suggests the need either to take higher moments into account, which substantially complicates the analysis, or to use measures of riskiness other than the standard deviation.

More fundamental criticisms arise from the work of Ellsberg (1961). Mean-variance analysis typically treats probabilities as if they are objectively known, or at least as if they can be derived from observed preferences as in Savage (1954). But there is ample evidence to suggest that many decisionmakers do not display preferences consistent with well-defined subjective probabilities (probabilistic sophistication in the terminology of Machina and Schmeidler 1992). In particular, preferences may display source dependence. Decisionmakers may prefer either side of a symmetric bet that is well understood (for example a coin toss) over either side of an apparently symmetric

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<sup>1</sup> For example, expected utility with quadratic utility implies the risk preferences exhibit increasing absolute risk aversion.

bet on an unfamiliar event (for example, up or down daily movements in a foreign stock market).

In this paper, we address all of these issues. We provide a rigorous foundation for preferences characterized by two arguments, a mean value and a dispersion parameter. The properties of the dispersion parameter generalize those of the standard deviation and are satisfied (modulo an appropriate normalization in some cases) by many of the commonly used measures of dispersion in the statistical literature. Our approach, however, encompasses choice under risk (known objective probabilities), choice under uncertainty (subjective probabilities as in Savage) and choice under ambiguity (where different ‘sources’ of uncertainty need not be treated symmetrically).

We show that results analogous to those that hold for the standard CAPM as well as a wide range of comparative static results may be derived for this tractable model. The key insight is that many well behaved problems of decision under uncertainty can be reduced to a simple two-step procedure.

First, we show that the choice set is convex in mean-dispersion space. Given the additional hypothesis of an unconstrained allocation to a riskless option, we derive a linear frontier as in the two-fund separation analysis of Sharpe (1964). In the second stage, given standard (strict) convexity conditions, we show that the (unique) optimal decision arises at the point of tangency between the choice set and the mean-dispersion trade-off. Standard comparative static results, which can be illustrated in the familiar graphical setup, are therefore applicable. In addition we show in a finance setting a two-fund decomposition result holds and, furthermore, derive a CAPM style asset pricing formula.

## 2 Background

In Grant and Polak (2011) two of the coauthors of the present paper examine the family of mean-dispersion preferences that admit a representation that takes the form of a ‘mean’ minus a ‘dispersion measure’ of the state-contingent utility vector associated with an act. In particular, for these preferences, one can show that there exists an affine utility function  $U$  over consequences, a probability weighting vector  $\pi$  on the states and a function  $\rho$  over state-utility vectors such that

the preferences over acts are represented by the functional

$$V(f) = E_\pi(U \circ f) - \rho(U \circ f), \quad (1)$$

where  $U \circ f$  is the utility vector given by  $(U \circ f)_s := U(f(s))$  and  $E_\pi(u) := \sum_s \pi_s u_s$ , for each utility vector  $u$ . Moreover,  $\rho(0) = 0$  and  $\rho$  exhibits translation invariance in the sense that, letting  $e$  denote the constant state-utility vector  $(1, \dots, 1)$ , we have  $\rho(u + \delta e) = \rho(u)$ , for all  $\delta$ .

For the case where  $\rho(u) \geq 0$ , we can view  $\rho(u)$  as a measure of dispersion of the utility vector  $u$ . The interpretation is that the agent with these preferences dislikes dispersion. More specifically, for each act  $f$ , let  $x_f$  be a constant act such that  $x_f \sim f$ . Then, the measure of dispersion  $\rho(U \circ f)$  associated with the act  $f$  is given by  $E_\pi(U \circ f) - U(x_f)$ : it is the reduction in expected utility the agent would be willing to accept in return for removing all the state-contingent utility uncertainty associated with the act. Drawing an analogy from choice under risk, we can think of  $x_f$  as corresponding to a certainty equivalent and of  $\rho(U \circ f)$  as corresponding to an absolute risk premium. Thus,  $\rho(U \circ f)$  is an ‘‘absolute uncertainty premium’’.<sup>2</sup> Since  $\rho$  exhibits translation invariance and  $V$  is linear in  $\rho$ , mean-dispersion preferences exhibit the property of *constant absolute uncertainty aversion*.

Siniscalchi (2009) characterizes an important special case of such preferences which he calls vector expected utility preferences. In his model, however,  $\rho$  satisfies ‘complementary symmetry’. Essentially complementary symmetry entails that for any utility vector  $u$  we require  $\rho(u) = \rho(-u)$ . Grant and Polak (2011) show that mean-dispersion preferences include the variational preferences of Maccheroni et al. (2006) (and thus also the multiple prior model of Gilboa and Schmeidler [1989] and the multiplier preferences of Hansen and Sargent [2001]).

The starting point for this paper is the observation that a constant uncertainty premium is a restrictive assumption. How plausible we find this restriction may depend on the stories we use to interpret these models. For example, one could interpret a multiple-prior set as simply reflecting the set of probabilities over states of the world that the agent perceives as possible. There is no reason for this perceived set to change as the agent becomes better off, and so, in this interpreta-

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<sup>2</sup> This ‘uncertainty premium’ is the amount that the agent is willing to pay to ensure that she obtains the same utility in every state. This premium corresponds to the notion of ‘ambiguity’ aversion in Ghirardato & Marinacci (2002).

tion, a constant uncertainty premium is perhaps quite plausible. But an alternative interpretation of mean dispersion preferences (even in the multiple-prior case) is that they reflect not just the agent’s perceptions of ambiguity but also the agent’s dislike of any perceived ambiguity.<sup>3</sup> Indeed, the term ‘ambiguity averse’ seems to suggest dislike rather than just perception. If we believe this dislike-of-ambiguity interpretation then it seems less plausible that uncertainty premia should be constant: just as agents with higher mean wealth tend to care less about a given monetary risk, so agents with higher mean utility might tend to care less about a given state-contingent dispersion of utility.

With this in mind, we develop a model that allows uncertainty premia to vary as we change mean utility. The new model maintains the tractable feature that preferences can be expressed in terms of two summary statistics: a mean and a measure of dispersion of the state utility vector. To aid tractability in applications, the family of preferences we characterize admit a representation in which the dispersion measure exhibits positive linear homogeneity, sub-additivity, translation invariance and complementary symmetry.<sup>4</sup> However, these preferences are only weakly separable in terms of the mean and the dispersion.

We call such preferences “invariant symmetric preferences”, and they assume the general form:

$$V(f) = \varphi(E_\pi(U \circ f), \rho(U \circ f)),$$

where  $\varphi$  is increasing in its first argument, nonincreasing in its second argument, and  $\varphi(y, 0) = y$ .

Using this latter property yields the following obvious, but informative, decomposition

$$\begin{aligned} V(f) &= E_\pi(U \circ f) - [E_\pi(U \circ f) - \varphi(E_\pi(U \circ f), \rho(U \circ f))] \\ &= E_\pi(U \circ f) - [\varphi(E_\pi(U \circ f), 0) - \varphi(E_\pi(U \circ f), \rho(U \circ f))]. \end{aligned}$$

That is, we may interpret the difference  $\varphi(\mu, 0) - \varphi(\mu, \rho)$  as the “absolute uncertainty premium”

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<sup>3</sup> For example, in one story used to interpret  $\varepsilon$ -perturbation models, the agent is thought of as perceiving a large set of possible probabilities (for example, the entire simplex) but then only putting weight  $\varepsilon$  on the ‘worst probability’ from this set with the remaining  $(1 - \varepsilon)$  on a particular prior. In this case, the ‘revealed-preference’ multiple-prior set is obviously much smaller than the entire simplex. If we believe the story, it is not obvious that the  $\varepsilon$  (and hence the revealed multiple-prior set) should remain constant as the agent becomes better off.

<sup>4</sup> Together the first two properties imply convexity. The first three characterize the finite state space analog of Rockafellar et al’s (2006) *general deviation measure*. All four properties are ones that typically hold for well-known measures of dispersion in the statistics literature, such as standard deviation, mean absolute deviation and Gini’s mean difference.

(measured in ‘utils’) of an (and any) act with mean utility  $\mu$  and measure of dispersion  $\rho$ .

The closest analog of this model in the context of risk preferences are the invariant risk preferences introduced by Quiggin & Chambers (2004). Indeed, we chose the name ‘invariant symmetric’ since we view this class of preferences as an analog of Quiggin and Chambers’ invariant risk preferences for the setting of subjective uncertainty, albeit with a symmetric (but not necessarily [second-order] probabilistically sophisticated) dispersion measure playing the role of their risk measure. Furthermore, since we are in a setting of subjective uncertainty, in our model the probabilities over the states are not given exogenously but rather are derived as part of the representation from purely behavioral properties of the underlying preferences.

In what follows, we provide an axiomatization of the invariant symmetric preferences model. Our axioms utilize a key axiom from Siniscalchi’s axiomatization of vector expected utility preferences. Like Maccheroni et al (2006) we weaken Gilboa and Schmeidler’s (1989) constant independence axiom which itself was a weakening of Anscombe-Aumann’s (1963) independence axiom. Thus, the standard subjective expected utility model is nested in our axiomatization.

An example may aid intuition. Standard deviation is a natural candidate for a measure of dispersion. The standard mean-variance model is linear in the mean and the square of the standard deviation, thus exhibiting constant absolute risk aversion. Epstein (1985) introduced a more general mean-variance model precisely to capture the idea of decreasing absolute risk aversion. His mean-variance functionals are weakly separable in the mean and the standard deviation, just as in expression 1. Thus, we can think of Epstein’s model as an example of our more general invariant symmetric preferences.

Section 4 introduces the main axioms and main representation theorem for our more general mean-dispersion preferences. We show in sections 5 and 6 how the model can accommodate non-constant uncertainty aversion. In section 7 we provide comparative static results for a canonical asset allocation problem and show that if the preferences admit an invariant symmetric representation with linear utility then a two-fund separation result applies.

### 3 Invariant Symmetric Preferences.

Our set-up is similar to Maccheroni et al.'s (2006) except we take the state space  $S$  to be a finite set  $\{s_1, \dots, s_n\}$  denoting the possible states of nature that may obtain. Let  $X$  be the set of consequences. An act is a function  $f : S \rightarrow X$ . With slight abuse of notation, any  $x$  in  $X$  will also denote the constant act that yields  $x$  in every state. Let  $\mathcal{F}$  denote the set of acts and continuing our abuse of notation,  $X$  shall also denote the set of constant acts. In addition we shall assume  $X$  is a convex subset of a vector space. For example, in Fishburn's (1970) rendition of the Anscombe-Aumann (1963) setting,  $X$  is taken to be the set of all lotteries on a set of final prizes. Alternatively, in finance applications  $X$  is often taken to be a subset of the positive reals. This means both the sets  $X$  and  $\mathcal{F}$  are mixture spaces. In particular, for any pair of acts  $f$  and  $g$  in  $\mathcal{F}$ , and any  $\alpha$  in  $(0, 1)$ , we take  $\alpha f + (1 - \alpha)g$  to be the act  $h \in \mathcal{F}$ , in which  $h(s) = \alpha f(s) + (1 - \alpha)g(s) \in X$ , for each  $s$  in  $S$ .

The decision maker's preferences on  $\mathcal{F}$  are given by a binary relation  $\succsim$ . Let  $\succ$  denote the strict preference and  $\sim$  denote indifference derived from  $\succsim$  in the usual way.

**Definition 1** *For all acts  $f$  in  $\mathcal{F}$ , we say that a constant act  $x_f$  in  $X$  is a certainty equivalent of  $f$  if  $x_f \sim f$ .*

For most models dealing with a mixture space of acts, the first step is to show that the axioms induce an expected utility representation over the set of constant acts. That is, there exists an affine function  $U : X \rightarrow \mathbb{R}$ , that represents  $\succsim$  restricted to  $X$ . Affinity of  $U$  means that  $U(\alpha x + (1 - \alpha)y) = \alpha U(x) + (1 - \alpha)U(y)$ , for all  $x$  and  $y$  in  $X$ , and all  $\alpha$  in  $[0, 1]$ .

Once we have obtained a utility representation of the preferences on the constant acts, it is convenient to map each act to its corresponding state-utility vector, and to consider the preference relation over these state-utility vectors induced by the underlying preferences over acts.

Thus, given an affine function  $U : X \rightarrow \mathbb{R}$  on the constant acts, where  $0$  is in the interior of  $U(X)$ , we can map each act  $f$  to the state-utility vector  $U \circ f \in U(X)^n$  given by  $(U \circ f)_s = U(f(s))$ . Recalling from the previous section that  $e$  denotes the constant vector  $(1, \dots, 1) \in \mathbb{R}^n$ , we will refer to the set  $\{ke : k \in U(X)\}$  as the constant-utility line. For any given  $U$ , constant



acts are mapped to state-utility vectors that lie on the constant-utility line.

The preferences  $\succsim$  induce preferences on the state-utility vectors in  $U(X)^n$  in the natural way.

**Definition 2 (Induced Preferences).** Let  $\succsim_u$  be the binary relation on  $U(X)^n$  defined by:  $u \succsim_u u'$  if there exists a corresponding pair of acts  $f$  and  $f'$  in  $\mathcal{F}$  with  $U \circ f = u$  and  $U \circ f' = u'$ , such that  $f \succsim f'$ .

Let  $\Delta(S)$  denote the set of probability measures over  $S$ . For each  $\pi \in \Delta(S)$ , let  $\pi_s := \pi(\{s\})$  for each  $s \in S$  and let  $E_\pi(u) := \sum_s \pi_s u_s$ , for each  $u \in \mathbb{R}^n$ . We will often refer to  $E_\pi(u)$  as a mean utility (of  $u$  with respect to  $\pi$ ).

We can now formally define the family of invariant symmetric preferences.

**Definition 3** An invariant symmetric representation is a tuple  $\langle U, \pi, \rho, \varphi \rangle$  where:

1.  $U : X \rightarrow \mathbb{R}$  is an affine utility function with 0 in the interior of the range;
2.  $\pi \in \Delta(S)$  is a baseline probability;
3.  $\rho : U(X)^n \rightarrow \mathbb{R}$  is a continuous, dispersion measure with
  - (a)  $\rho(\lambda u) = \lambda \rho(u) \geq 0$ , for all  $u$  in  $U(X)^n$  and all  $\lambda \geq 0$ , such that  $\lambda u$  is also in  $U(X)^n$  (positive linear homogeneity),
  - (b)  $\rho(u + u') \leq \rho(u) + \rho(u')$ , for all  $u$  and  $u'$  in  $U(X)^n$  such that  $(u + u')$  is also in  $U(X)^n$  (sub-additivity),
  - (c)  $\rho(u + \delta e) = \rho(u)$ , for all  $u$  in  $U(X)^n$  and all  $\delta$  in  $\mathbb{R}$  such that  $u + \delta e$  is also in  $U(X)^n$  (translation invariance), and
  - (d)  $\rho(u) = \rho(-u)$ , for all  $u$  in  $U(X)^n$  such that  $-u$  is also in  $U(X)^n$  (symmetry); and
4.  $\varphi : \mathcal{D} \rightarrow \mathbb{R}$ , is a continuous function, with domain  $\mathcal{D} \subset U(X) \times \rho(U(X)^n)$  comprising pairs  $(\mu', \rho')$  for which  $\mu' = E_\pi(u)$  for some  $u \in \rho^{-1}(\rho')$ , increasing in its first argument, non-increasing in its second argument, with  $\varphi(y, 0) = y$  for all  $y$  in  $U(X)$ , and monotonic in  $u$ , that is,

$$\varphi(E_\pi(u), \rho(u)) - \varphi(E_\pi(u'), \rho(u')) \geq 0,$$

for all  $u \geq u'$  in  $U(X)^n$ .

The associated invariant symmetric preferences over acts are generated by

$$V(f) = \varphi(E_\pi(U \circ f), \rho(U \circ f)), \quad (2)$$

where  $U \circ f$  is the utility vector given by  $(U \circ f)_s := U(f(s))$ .

An invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$  is labeled compact if  $U(X)$  is compact (that is, closed and bounded).

In this representation,  $E_\pi(U \circ f)$  represents the “mean utility” of the act  $f$  using the utility function  $U(\cdot)$  and the weights  $\pi$ . We can think of  $\rho(U \circ f)$  as a measure of dispersion of the state-utility vector  $U \circ f$ . The overall representation is weakly separable in the mean and dispersion, and is strictly increasing in the former and non-increasing in the latter. The normalization  $\varphi(y, 0) = y$  ensures that the value of a constant act  $x$  is equal to the utility of that act,  $V(x) = U(x)$ , and hence that the value of a general act  $f$  is equal to the utility of its certainty equivalent,  $V(f) = U(x_f)$ .

### 3.1 Uniqueness

For preferences that admit the invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ , the base-line probability  $\pi$  is unique but there is some indeterminacy in specifying the other three components of the representation. Not surprisingly, the utility function  $U$  is unique only up to a positive affine transformation, while the measure of dispersion is unique up to multiplication by a positive scalar. That is, we can take a positive affine transformation of the utility function and a positive multiple of the suitably adjusted measure of dispersion function and then with appropriate adjustments to  $\varphi$  obtain another invariant symmetric representation for the same preferences.

We state the precise class of invariant symmetric representations that generate the same preferences in the following lemma.

**Lemma 1 (State-Utility Preferences)** *Fix an invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$  and let  $\succsim$  be the preferences generated by  $\langle U, \pi, \rho, \varphi \rangle$ . For any  $\alpha, \gamma > 0$  and any  $\beta \in \mathbb{R}$ , if*

$\tilde{U}(x) := \alpha U(x) + \beta$ ,  $\tilde{\rho}(u') := \gamma \rho([u' - \beta e] / \alpha)$  and  $\tilde{\varphi}(\mu', \rho') := \alpha \varphi([\mu' - \beta] / \alpha, \rho' / \gamma) + \beta$ , then  $\langle \tilde{U}, \pi, \tilde{\rho}, \tilde{\varphi} \rangle$  is an invariant symmetric representation of  $\succsim$ .

To see this notice that for the representation of  $\succsim$  corresponding to  $\langle \tilde{U}, \pi, \tilde{\rho}, \tilde{\varphi} \rangle$  we have:

$$\begin{aligned} \tilde{V}(f) &= \tilde{\varphi}\left(E_\pi(\tilde{U} \circ f), \tilde{\rho}(\tilde{U} \circ f)\right) \\ &= \alpha \varphi\left([E_\pi(\alpha U \circ f + \beta e) - \beta] / \alpha, \gamma \rho([\alpha U \circ f + \beta e - \beta e] / \alpha) / \gamma\right) + \beta \\ &= \alpha \varphi(E_\pi(U \circ f), \rho(U \circ f)) + \beta = \alpha V(f) + \beta. \end{aligned}$$

In order to pin down these components for a preference relation that admits a compact invariant symmetric representation we propose taking the element of the class of invariant symmetric representations for these preferences that has a normalized utility centered around 0, and given this normalized utility has a ‘maximal’ (in a sense we define below) measure of dispersion.

To define this *canonical* member of the class of invariant symmetric representations, given the invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ , we first set

$$\bar{U}(x) := \bar{\alpha}U(x) + \bar{\beta}, \text{ where } \bar{\alpha} = \frac{2}{\max_{x \in X} U(x) - \min_{x \in X} U(x)} \ \& \ \bar{\beta} = -\frac{[\max_{x \in X} U(x) + \min_{x \in X} U(x)]}{\max_{x \in X} U(x) - \min_{x \in X} U(x)}.$$

By construction  $\max_{x \in X} \bar{U}(z) = 1$  and  $\min_{x \in X} \bar{U}(w) = -1$ , thus making the range  $\bar{U}(X)$  equal to  $[-1, 1]$  which is indeed symmetric around 0. This in turn allows us to have a normalized domain that is symmetric around  $0e$  for the ‘maximal’ measure of dispersion.

To derive the ‘maximal’ measure of dispersion consider the family of risk measures defined on the domain of utility vectors  $[-1, 1]^n$  associated with a positive scalar multiple of  $\rho$  obtained by subtracting the mean utility. More precisely, for each  $\gamma \geq 0$ , let  $r_\gamma : [-1, 1]^n \rightarrow \mathbb{R}$  be the risk measure given by:

$$r_\gamma(u') := \rho_\gamma(u') - E_\pi(u'), \text{ where } \rho_\gamma(u') := \gamma \rho([u' - \bar{\beta}e] / \bar{\alpha})$$

By construction, for each  $\gamma \geq 0$ ,  $r_\gamma$  is positive linear homogeneous, sub-additive and translation invariant.<sup>5</sup> The risk measure  $r_\gamma$  is coherent if in addition it is weakly decreasing, that is  $u \geq u'$

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<sup>5</sup> In the context of risk measures, translation invariance is the property that  $r_\gamma(u + \delta e) = r_\gamma(u) - \delta$ , for all  $u$  in  $[-1, 1]^n$  and all  $\delta$  in  $\mathbb{R}$  such that  $u + \delta e$  is also in  $[-1, 1]^n$ .

implies  $r_\gamma(u) \leq r_\gamma(u')$ .<sup>6</sup>

Notice that elements of the sets  $\{r_\gamma : \gamma \geq 0\}$  and  $\{\rho_\gamma : \gamma \geq 0\}$  are ordered according to  $\gamma$ . That is,  $\gamma' > \gamma''$  implies  $r_{\gamma'}(u) \geq r_{\gamma''}(u)$  and  $\rho_{\gamma'}(u) \geq \rho_{\gamma''}(u)$ , for all  $u$  in  $[0, 1]^n$ .

Notice that  $r_0(u) \equiv -E_\pi(u)$  is decreasing in  $u$ . Notice also that, for any  $u$  and  $u'$  in  $[-1, 1]^n$  such that  $u \geq u'$ ,  $u \neq u'$  and  $\rho(u) > \rho(u')$ , there exists  $\gamma'(u, u')$ , given by,

$$\gamma'(u, u') = \frac{E_\pi(u) - E_\pi(u')}{\rho([u - \bar{\beta}e] / \bar{\alpha}) - \rho([u' - \bar{\beta}e] / \bar{\alpha})}.$$

From this it follows that for any  $\gamma < \gamma'(u, u')$ :

$$[r_\gamma(u) - r_\gamma(u')] < [\rho_{\gamma'(u, u')}(u') - \rho_{\gamma'(u, u')}(u)] - [E_\pi(u) - E_\pi(u')] = 0,$$

and any  $\gamma > \gamma'(u, u')$ :

$$[r_\gamma(u) - r_\gamma(u')] > [\rho_{\gamma'(u, u')}(u') - \rho_{\gamma'(u, u')}(u)] - [E_\pi(u) - E_\pi(u')] = 0,$$

That is, for  $\gamma > \gamma'(u, u')$ , the risk measure  $r_\gamma$  is not weakly decreasing and hence is not coherent.

Our candidate for the canonical measure of dispersion is the maximal one from the set  $\{\rho_\gamma : \gamma \geq 0\}$  for which the associated risk measure is weakly decreasing (and hence coherent).

Formally, let

$$\bar{\gamma} = \inf \{\gamma'(u, u') : u \geq u', u \neq u', \text{ and } \rho(u) > \rho(u')\}.$$

For any  $\gamma \leq \bar{\gamma}$ ,  $r_\gamma$  is weakly decreasing, and for any  $\gamma > \bar{\gamma}$ ,  $r_\gamma$  is not weakly decreasing. Thus, we may define the canonical risk measure by setting  $\bar{\rho}(u) := \rho_{\bar{\gamma}}(u)$ , and hence set  $\bar{\varphi}(\mu', \rho') := \bar{\alpha}\varphi([ \mu' - \bar{\beta}e ] / \bar{\alpha}, \rho' / \bar{\gamma}) + \bar{\beta}$  to obtain the canonical representation  $\langle \bar{U}, \pi, \bar{\rho}, \bar{\varphi} \rangle$ . For a given invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ , the uniqueness of the associated canonical representation  $\langle \bar{U}, \pi, \bar{\rho}, \bar{\varphi} \rangle$  is immediate from its construction as detailed above. We summarize this with the following definition and proposition.

**Definition 4** *An invariant symmetric representation  $\langle \bar{U}, \pi, \bar{\rho}, \bar{\varphi} \rangle$  is canonical if  $\bar{U}(X)$  is the interval  $[-1, 1]$ , the associated risk measure  $\bar{r}(u) := \bar{\rho}(u) - E_\pi(u)$  is weakly decreasing and for any  $\gamma > 1$ , the risk measure  $r_\gamma(u) := \gamma\bar{\rho}(u) - E_\pi(u)$  is not.*

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<sup>6</sup> For the definition of coherent risk measures see for example Artzner et al (1999).

**Proposition 2 (Uniqueness of Canonical Representation)** *Suppose  $\succsim$  admits a compact invariant symmetric representation. Then there exists a unique canonical invariant symmetric representation that represents  $\succsim$ .*

Notice that for the canonical invariant symmetric representation  $\langle \bar{U}, \pi, \bar{\rho}, \bar{\varphi} \rangle$ , for every act  $f$  in  $\mathcal{F}$ ,

$$E_{\pi}(\bar{U} \circ f) \geq V(f) \geq E_{\pi}(\bar{U} \circ f) - \bar{\rho}(\bar{U} \circ f) \quad (= -\bar{r}(\bar{U} \circ f)).$$

That is, the mean utility and the negative of the associated risk measure of the act provide upper and lower bounds for the invariant symmetric representation, independent of the aggregator function  $\bar{\varphi}$ . Moreover, if two canonical representations share the same  $\bar{U}$ ,  $\pi$  and  $\bar{\rho}$  (that is, the same risk preferences, baseline measure and [maximal] measure of dispersion) then the differences in their attitudes toward trade-offs between expected return and dispersion will be reflected in the differences between their respective  $\bar{\varphi}$  functions.

## 4 Axioms and Main Theorem.

The first two axioms below, an ordering and a continuity axiom are standard.

**A.1 Order.**  $\succsim$  is transitive and complete.

**A.2 Continuity.** For any three acts  $f, g$  and  $h$  in  $\mathcal{F}$  such that  $f \succsim h \succsim g$ , the sets  $\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h\}$  and  $\{\alpha \in [0, 1] : h \succsim \alpha f + (1 - \alpha)g\}$  are closed.

It is also usual to have some form of monotonicity axiom that also delivers state independence and to have a non-degeneracy axiom.

**A.3 Monotonicity.** For any pair of acts  $f$  and  $g$  in  $\mathcal{F}$ , if  $f(s) \succsim g(s)$  for all  $s \in S$ , then  $f \succsim g$ .

**A.4 Best and Worst Outcome.** There exists outcomes  $z$  and  $w$  in  $X$  satisfying  $z \succ w$  and  $z \succ f \succ w$ , for all  $f$  in  $\mathcal{F}$ .

The stronger axiom A.4 assuming the existence of a best and a worst outcome is not essential in what follows but it simplifies the analysis ensuring that the representation obtained is bounded above and below.

The next axiom builds on Siniscalchi’s (2009) notion of ‘complementary acts’.

**Definition 5** *Two acts  $f$  and  $\bar{f}$  are complementary if  $\frac{1}{2}f + \frac{1}{2}\bar{f} = x$  for some  $x$  in  $X$ .*

*If two acts  $f$  and  $\bar{f}$  are complementary then  $(f, \bar{f})$  is referred to as a complementary pair.*

Notice that two acts are complementary if a fifty-fifty mixture of the pair provides a ‘perfect’ hedge against subjective uncertainty.<sup>7</sup> Furthermore, if preferences over constant acts admit the expected utility representation  $U$ , then the state-utility vectors associated with the complementary pair of acts  $(f, \bar{f})$ , denoted by  $U \circ f$  and  $U \circ \bar{f}$  satisfy  $U \circ f = 2ke - U \circ \bar{f}$  (or equivalently,  $\frac{1}{2}U \circ f + \frac{1}{2}U \circ \bar{f} = ke$ ) for some constant  $k \in U(X)$ . “Thus, complementarity is the preference counterpart of algebraic negation.” (Siniscalchi [2009, p. 810]) And, if  $(f, \bar{f})$  and  $(g, \bar{g})$  are complementary pairs of acts, with  $\frac{1}{2}f + \frac{1}{2}\bar{f} = x$  and  $\frac{1}{2}g + \frac{1}{2}\bar{g} = y$ , then, for any weight  $\alpha$  in  $[0, 1]$ , the pair  $(\alpha f + (1 - \alpha)g, \alpha \bar{f} + (1 - \alpha)\bar{g})$  is also a complementary pair, since

$$\begin{aligned} & \frac{1}{2}(\alpha f + (1 - \alpha)g) + \frac{1}{2}(\alpha \bar{f} + (1 - \alpha)\bar{g}) \\ &= \alpha \left[ \frac{1}{2}f + \frac{1}{2}\bar{f} \right] + (1 - \alpha) \left[ \frac{1}{2}g + \frac{1}{2}\bar{g} \right] = \alpha x + (1 - \alpha)y. \end{aligned}$$

As the state-utility vectors associated with a pair of complementary acts are reflections of each other in the constant-utility line “mirror”, all symmetric measures of dispersion attribute to them the same utility variability. So, if we are attributing the same utility variability to any pair of complementary acts, then plausibly we might rank such pairs of acts according to the same baseline measure. Hence, if a given pair of complementary acts  $(f, \bar{f})$  are indifferent to each other, then this suggests that those two acts have the *same* mean utility according to the baseline measure which in turn is the utility of any constant act that is indifferent to the perfect hedge  $\frac{1}{2}f + \frac{1}{2}\bar{f}$ . This is illustrated in figure 1 which plots the state-utility vectors of two complementary acts  $f$  and  $\bar{f}$  that are also indifferent to each other.

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<sup>7</sup> Siniscalchi’s definition differs from ours. He defines as complementary any pair of acts for which a fifty-fifty mixture constitutes a subjectively perfect hedge. Formally, for him any acts  $f$  and  $\bar{f}$  are *complementary* if, for any two states  $s$  and  $s'$ ,

$$\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s').$$

The advantage of our definition is that it is ‘preference’ free. Any pair of acts which are complementary under our definition are complementary for any decision maker. But given A.3 monotonicity and A.4 (existence of best and worst outcome), the set of acts that map to lotteries whose support is a subset of the best and worst outcomes are rich enough to provide enough pairs of complementary acts that enable us to derive the same implications as Siniscalchi achieved with his preference-based definition.

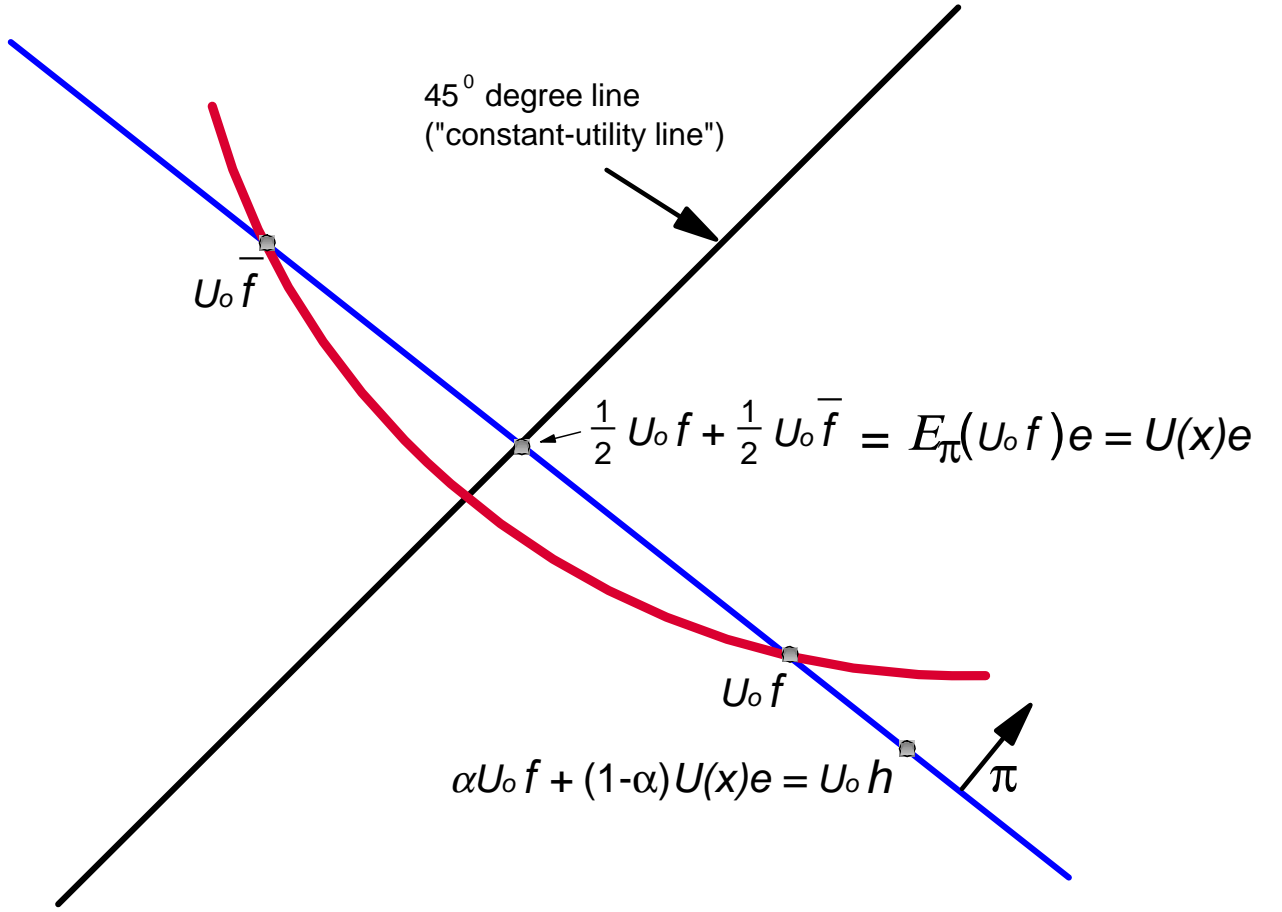


Figure 1: An illustration of the two state-utility vectors associated with two complementary acts  $f$  and  $\bar{f}$  that are also indifferent to each other.

The next axiom, Complementary Independence (due to Siniscalchi, 2009), formalizes this intuition that the ranking of pairs of complementary should be in accordance with their expectations taken with respect to the underlying baseline measure and hence should conform to expected utility theory.

**A.5 Complementary independence.** For any two complementary pairs of acts  $(f, \bar{f}), (g, \bar{g})$  : if

$$f \succsim \bar{f} \text{ and } g \succsim \bar{g} \text{ then } \alpha f + (1 - \alpha) g \succsim \alpha \bar{f} + (1 - \alpha) \bar{g} \text{ for all } \alpha \text{ in } (0, 1).$$

If we deem the constant act  $x$  to be the mean of the act  $f$  because there exists another act  $\bar{f}$  that is both complementary and indifferent to  $f$  and  $x = \frac{1}{2}f + \frac{1}{2}\bar{f}$ , then it seems natural to

consider  $x$  to be the mean of any other act  $h$  in which  $h = \lambda f + (1 - \lambda)x$  for some  $\lambda$  in  $(0, 1]$ . As we see in figure 1 the plot of the state-utility vector associated with such an act  $h$  lies on the ray from  $U(x)e$  through  $U \circ f$  and hence resides in the hyperplane through  $U(x)e$  with normal vector  $\pi$ . This motivates the following notion of the mean (constant) act for an act that is defined directly in terms of the underlying preferences over acts.

**Definition 6 (Mean and Common Mean)** *A constant act  $x$  is deemed to be the mean for an act  $f$  if there exists an act  $f'$  and  $\lambda \in (0, 1]$ , such that  $\lambda f + (1 - \lambda)x \sim f'$ , and  $\frac{1}{2}[\lambda f + (1 - \lambda)x] + \frac{1}{2}f' = x$  (hence  $(\lambda f + (1 - \lambda)x, f')$  constitute a complementary pair of acts). If the constant act  $x$  is the mean of both  $f$  and  $g$ , then  $f$  and  $g$  are said to have a common mean.<sup>8</sup>*

With this definition of the mean of an act in hand, we can now introduce the last two axioms which can readily be seen to be weakenings of Gilboa and Schmeidler's (1989) uncertainty aversion axiom and certainty independence axiom, respectively. Formally, they restrict the application of those two axioms to pairs of acts that have a common mean.

**A.6 Common-Mean Uncertainty Aversion.** For any pair of acts  $f$  and  $g$ , and any  $\alpha$  in  $(0, 1)$ , if  $f$  and  $g$  have a common mean and  $f \sim g$  then  $\alpha f + (1 - \alpha)g \succsim f$ .

**A.7 Common-Mean Certainty Independence.** For any pair of acts  $f, g$  in  $\mathcal{F}$ , any constant act  $x$  in  $X$  and any  $\alpha$  in  $(0, 1)$ : if  $f$  and  $g$  have a common mean then

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x.$$

We can now state our main result.

**Theorem 3 (Main Theorem)** *The preferences  $\succsim$  on  $\mathcal{F}$  satisfy A.1 (weak order), A.2 (continuity), A.3 (monotonicity), A.4 (best and worst outcome), A.5 (complementary independence), A.6 (common-mean uncertainty aversion) and A.7 (common-mean certainty independence), if and only if they admit a compact invariant symmetric representation  $\langle U, \pi, \varphi, \rho \rangle$ .*

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<sup>8</sup> If the range  $U(X)$  were unbounded, it would be enough to work with a simpler definition in which  $\lambda = 1$ . However, when dealing with a bounded range we can no longer ensure that for every utility vector  $u$  in  $U(X)^n$  there exists a complementary vector  $\bar{u}$  also in  $U(X)^n$ , and a complementary pair of acts  $(f, \bar{f})$ , satisfying  $u = U \circ f$ ,  $\bar{u} = U \circ \bar{f}$  and  $f \sim \bar{f}$ , thereby ensuring the existence of mean utility for every utility vector in  $U(X)^n$ .



The proof of the theorem appears in the appendix. The next section, however, provides some intuition as to how common-mean certainty independence leads to a representation that is weakly separable in the mean and dispersion of the associated state-utility vector.

## 5 Interpretation and Geometry

Given an affine utility function  $U$  on outcomes (and on constant acts) and probability weights  $\pi$  on the states, we may associate with the act  $f$  the state-utility vector  $U \circ f$  whose mean with respect to  $\pi$  is  $\mu := E_\pi(U \circ f)$ . Furthermore, we can think of the absolute uncertainty premium (measured in utility) of the act  $f$  as being the difference  $\mu - U(x_f)$  between the mean utility and the utility of the certainty equivalent. For the mean-dispersion preferences that were analyzed in Grant and Polak (2011), this premium was equal to the measure of dispersion  $\rho(U \circ f)$ . With invariant symmetric preferences, the premium is given by  $\varphi(\mu, 0) - \varphi(\mu, \rho(U \circ f))$ . Thus, the premium depends not only on the measure of dispersion  $\rho(U \circ f)$  but also on  $\varphi$  which in turn depends on the mean utility  $\mu$ .

Figures 2-4 illustrate how the key axiom, common-mean certainty independence, allows uncertainty premia to vary. They are drawn for the case where the induced preferences over state-utility vectors are smooth. Suppose  $f$  and  $g$  are a pair of acts with common mean  $x$  and which are indifferent to each other. The associated state-utility vectors  $U \circ f$  and  $U \circ g$  are plotted in figure 2. Since  $f$  and  $g$  have common mean  $x$ , the associated state utility vectors  $U \circ f$  and  $U \circ g$  must both lie in the hyperplane through  $\mu e$ , where  $\mu = U(x)$ . Let  $\pi$  denote its normal vector. The certainty equivalent constant utility vector corresponds to the point  $V(f)e$  where the indifference set of  $\succsim_u$  in which  $U \circ f$  and  $U \circ g$  both reside intersects the constant-utility line. The uncertainty premium (measured in utils) is given by  $\mu - \varphi(\mu, \rho(U \circ f))$ .

Now for fixed  $\alpha$  in  $(0, 1)$ , consider the two acts  $\alpha f + (1 - \alpha)x$  and  $\alpha g + (1 - \alpha)x$  formed by taking convex combinations of the common mean  $x$  with  $f$  and with  $g$ , respectively. Applying axiom A.7 it follows that the state-utility vectors  $\alpha U \circ f + (1 - \alpha)\mu e$  and  $\alpha U \circ g + (1 - \alpha)\mu e$  which are plotted in figure 3 must lie on the same indifference curve. As  $\alpha$  was arbitrary and the same applies for any pair of acts that have  $x$  as a common mean, this in turn means the indifference

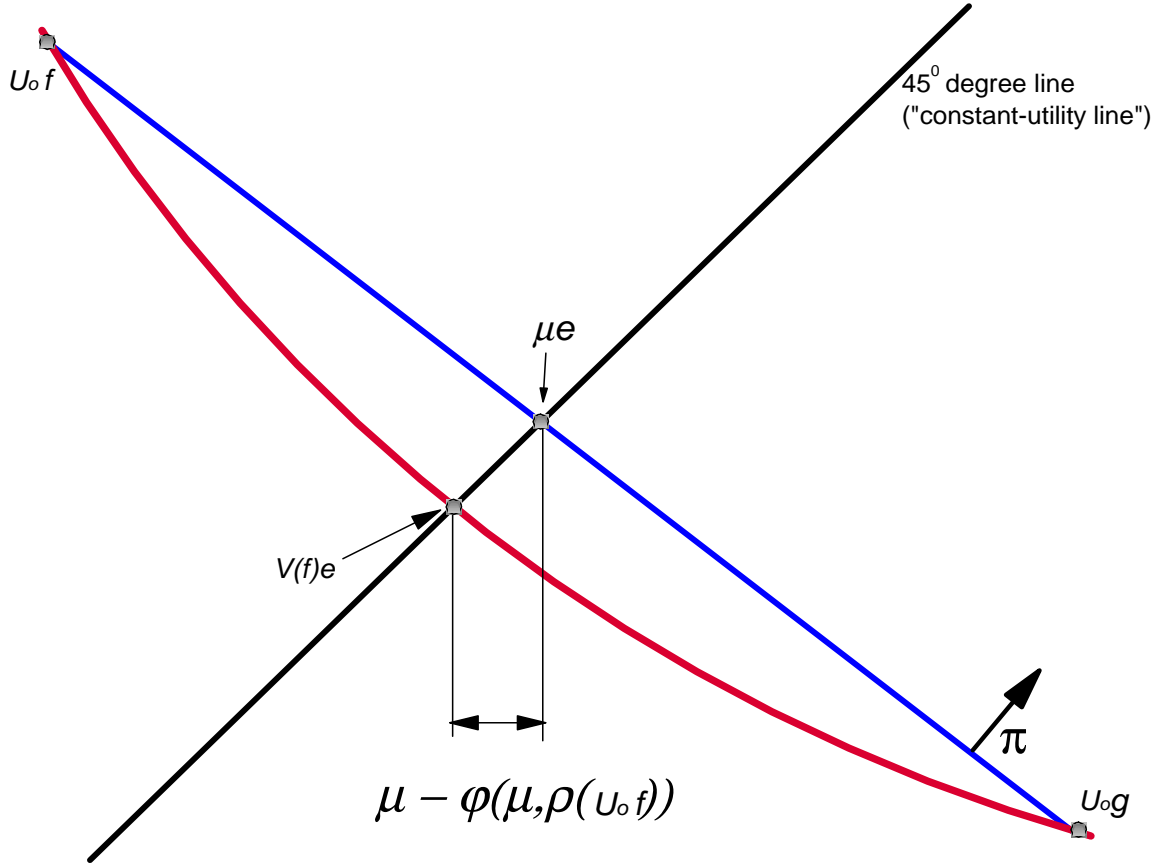


Figure 2: Illustration of uncertainty-premium.

map on the hyperplane through  $\mu e$  with normal vector  $\pi$  is *homothetic*.

More generally Lemma 16 in the Appendix implies the following scale invariance property of the induced preferences.

**Definition 7 (Common-Mean Radial Homotheticity.)** Fix  $\mu$  in  $U(X)$ . Suppose for any pair of complementary utility vectors  $(u, \bar{u})$  in  $U(X)^n$ , such that  $E_\pi(u) = E_\pi(\bar{u}) = \mu$ ,  $u \sim_u \bar{u}$ . Then for any pair of utility vectors  $u'$  and  $u''$  in  $U(X)^n$ , such that  $E_\pi(u') = E_\pi(u'') = \mu$  and any  $\alpha \in (0, 1)$ :  $u' \succeq_u u''$  if and only if  $\alpha u' + (1 - \alpha)\mu e \succeq_u \alpha u'' + (1 - \alpha)\mu e$ .

Next consider some other constant act  $y$  and the two new acts  $\alpha f + (1 - \alpha)y$  and  $\alpha g + (1 - \alpha)y$ , formed by taking convex combinations of  $y$  with  $f$  and with  $g$ , respectively. Once again, since

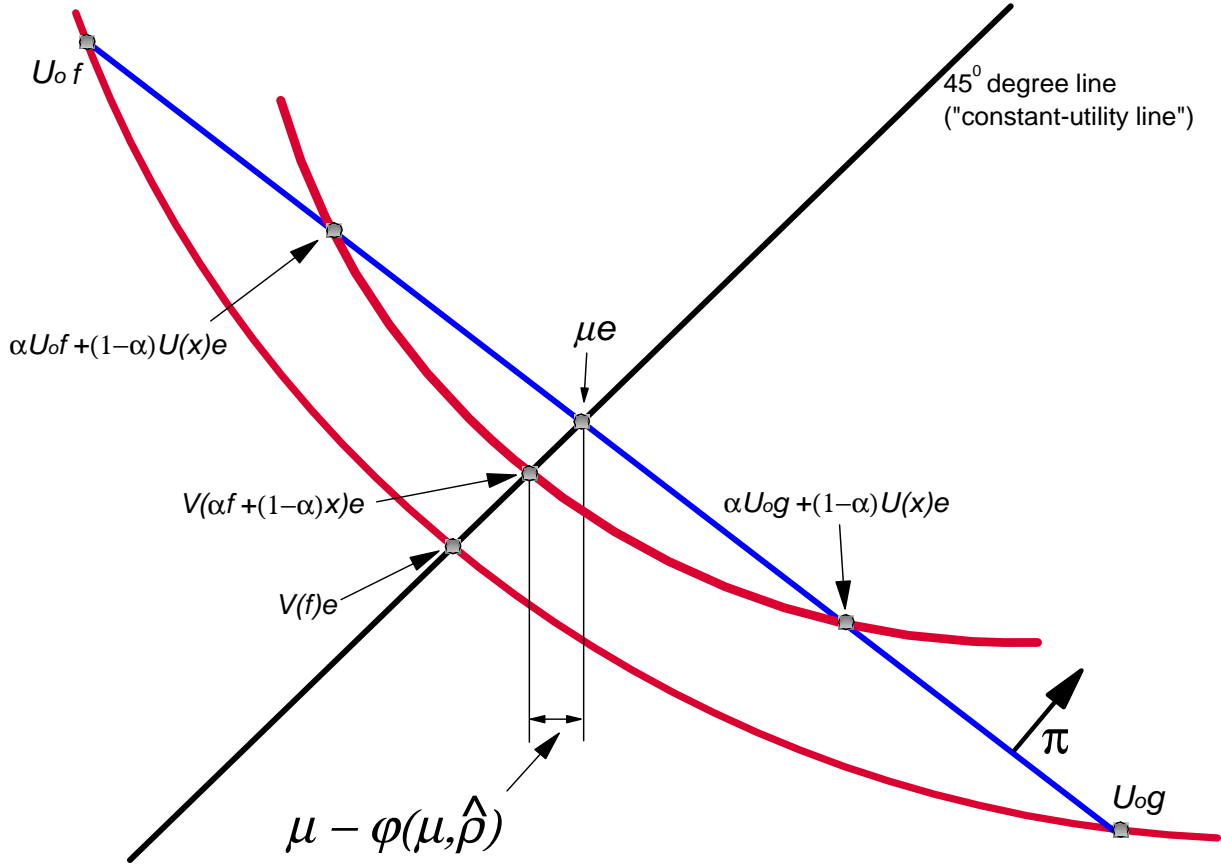


Figure 3: Illustration that indifference curves on an equal-mean hyperplane are homothetic.

$f$  and  $g$  have a common mean, we can apply axiom A.7. Hence the state-utility vectors  $\alpha U \circ f + (1 - \alpha) U(y)e$  and  $\alpha U \circ g + (1 - \alpha) U(y)e$  which are plotted in figure 4 must also lie on the same indifference curve. Again by construction the two new state-utility vectors  $\alpha U \circ f + (1 - \alpha) U(y)e$  and  $\alpha U \circ g + (1 - \alpha) U(y)e$  have the same mean with respect to  $\pi$ ; in this case  $\mu'$ . In fact, each of these two new indifferent vectors is obtained from the previous pair of indifferent state-contingent utility vectors by the common translation  $(1 - \alpha)(U(y) - U(x))e$  (that is, a translation parallel to the constant-utility line). More generally, lemma 17 in the appendix shows that axiom A.7 implies the following translation invariance property of the induced preferences  $\succsim_u$ .

**Definition 8 (Common-Mean Translation Invariance.)** Fix  $\mu$  in  $U(X)$ . Suppose for any pair of complementary utility vectors  $(u, \bar{u})$  in  $U(X)^n$ , such that  $E_\pi(u) = E_\pi(\bar{u}) = \mu$ ,  $u \sim_u \bar{u}$ .

Then for any pair of utility vectors  $u'$  and  $u''$  in  $U(X)^n$ , such that  $E_\pi(u') = E_\pi(u'') = \mu$  and any  $\delta \in \mathbb{R}$ , such that  $u' + \delta e$  and  $u'' + \delta e$  are both in  $U(X)^n$ ,  $u' \succsim_u u''$  if and only if  $u' + \delta e \succsim_u u'' + \delta e$ .

Although axiom A.7 is not weaker than Maccheroni et al.'s (2006) weak certainty independence axiom, the property of common-mean translation invariance is weaker than the translation invariance property implied by weak certainty independence. In particular there is no requirement that, if we apply the same common translation  $(1 - \alpha)(U(y) - U(x))e$  to the entire indifference curve through  $V(\alpha f + (1 - \alpha)x)e$ , then all points in the new translated curve will be indifferent. The reason is that not all the points on the original indifference curve had the same mean. In our picture, the actual indifference curve through  $\alpha U \circ f + (1 - \alpha)U(y)e$  is less bowed.

Now consider uncertainty premia. The mean of the two vectors  $\alpha f + (1 - \alpha)x$  and  $\alpha g + (1 - \alpha)x$  was  $\mu = \varphi(\mu, 0)$ . Since they had the same mean and were indifferent, they must have the same dispersion term  $\hat{\rho}$ : that is, the utility of their certainty equivalent is  $V(\alpha f + (1 - \alpha)x) = V(\alpha g + (1 - \alpha)x) = \varphi(\mu, \hat{\rho})$ . Thus the uncertainty premium associated with those two vectors is just  $\mu - \varphi(\mu, \hat{\rho})$ . The mean of the two vectors  $\alpha f + (1 - \alpha)y$  and  $\alpha g + (1 - \alpha)y$  was  $\mu'$ . By construction, they had the same vector of differences from this mean as the other two vectors, hence their dispersion term was also  $\hat{\rho}$ . Thus, the utility of their certainty equivalent  $V(\alpha f + (1 - \alpha)y) = V(\alpha g + (1 - \alpha)y) = \varphi(\mu', \hat{\rho})$ , and their uncertainty premium is just  $\mu' - \varphi(\mu', \hat{\rho})$ . But, as shown, these premia need not be the same: in the illustrated case, the uncertainty premium decreased as we increased the mean utility holding the dispersions fixed.

## 6 Absolute Uncertainty Aversion

In the analysis of risk, one way to define decreasing absolute risk aversion is (abusing our notation): for all random variables  $\tilde{X}, \tilde{Y}$  such that  $\tilde{X}$  is riskier than  $\tilde{Y}$  in some sense, if  $\tilde{X}$  is weakly preferred to  $\tilde{Y}$  then for any  $\delta > 0$ , the 'improved' random variable  $\tilde{X} + \delta$  is also weakly preferred to the improved random variable  $\tilde{Y} + \delta$ .<sup>9</sup> That is, the set of acceptable increases in risk can only expand and not contract as non-state contingent wealth is increased. Increasing absolute risk aversion can

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<sup>9</sup> Properly speaking this should be referred to as *non-increasing* absolute risk aversion, but we will follow the common usage in the risk literature.

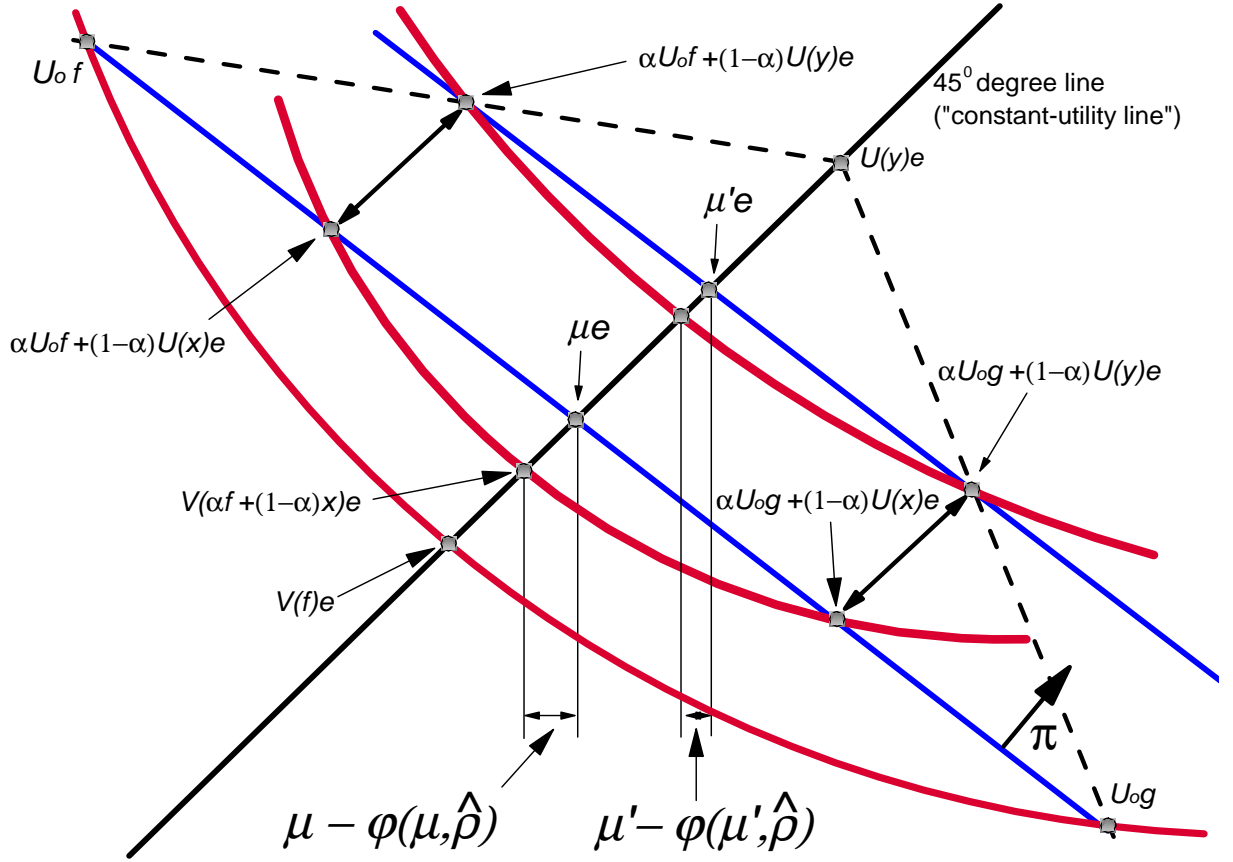


Figure 4: Illustration of invariant complementary-symmetric preferences with non-constant absolute uncertainty premium.

be defined similarly.

For the family of preferences defined over subjectively uncertain acts considered here, we can define analogous concepts of decreasing absolute uncertainty aversion and increasing absolute uncertainty aversion.

We begin by proposing one notion of what it might mean for one act to be deemed ‘more dispersed’ than another. In particular, we shall propose that if one act can be expressed as a convex combination of another act and a constant act then the latter act is deemed more dispersed than the former, since the former act is ‘between’ (in terms of mixtures) between the constant act (that by definition has zero dispersion) and the latter act.

**Definition 9 (“At least as dispersed as” Partial Ordering)** An act  $f$  is considered at least as dispersed as the act  $g$ , denoted  $f \succeq g$ , if there exists a constant act  $x$  and a  $\lambda \in [0, 1]$ , such that  $g = \lambda f + (1 - \lambda)x$ .

The relation  $\succeq$  respects Gilboa and Schmeidler’s Certainty Independence axiom.

**Proposition 4** For any pair of acts  $f$  and  $g$  in  $\mathcal{F}$ , any constant act  $y$  in  $X$  and any  $\alpha$  in  $(0, 1)$ :  $f \succeq g$  if and only if  $\alpha f + (1 - \alpha)y \succeq \alpha g + (1 - \alpha)y$ .

With the *at least as dispersed (partial) ordering*  $\succeq$  in hand, we can now define the corresponding notions of decreasing, increasing and constant absolute uncertainty aversion.

**Definition 10 (DAUA, IAUA and CAUA)** We say that  $\succsim$  exhibits decreasing absolute uncertainty aversion (DAUA) if, for any pair of acts  $f$  and  $g$  in  $\mathcal{F}$ , such that  $f \succeq g$ , any pair of constant acts  $x$  and  $y$ , such that  $y \succsim x$ , and any  $\alpha$  in  $(0, 1)$ :

$$\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x \Rightarrow \alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y. \quad (3)$$

We say the agent exhibits increasing absolute uncertainty aversion (IAUA) if expression (3) holds for any constant acts  $x$  and  $y$  such that  $x \succsim y$ . And we say the agent exhibits constant absolute uncertainty aversion (CAUA) if she exhibits both DAUA and IAUA.

The following proposition characterizes the class of invariant symmetric preferences that exhibit DAUA (respectively, IAUA).

**Proposition 5 (DAUA)** Suppose that the preferences  $\succsim$  admit the invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ . Then the following two properties are equivalent:

- (a) The preferences  $\succsim$  exhibit DAUA (respectively, IAUA)
- (b) Whenever  $\varphi(\mu, \rho) = \varphi(\mu', \rho')$  and  $\mu > \mu'$  then  $\varphi(\mu + \delta, \rho) \geq$  (resp.  $\leq$ )  $\varphi(\mu' + \delta, \rho')$  for all  $\delta > 0$ , such that for some  $\bar{u}'$  in  $\rho^{-1}(\rho')$ ,  $E_\pi(\bar{u}') = \mu' + \delta$ .

Furthermore, assuming  $\varphi$  is twice differentiable, (b) is equivalent to

$$\frac{-\varphi_{11}}{\varphi_1} \leq \text{(resp. } \geq) \frac{\varphi_{12}}{-\varphi_2}. \quad (4)$$

The left-hand side of inequality (4) resembles the Arrow-Pratt coefficient of absolute risk aversion from expected utility theory and measures the concavity of  $\varphi$  with respect to its first argument,  $\mu$ . It is also the negative of the semi-elasticity of  $\varphi_1$  with respect to  $\mu$ . Similarly, the right-hand side of inequality (4) is the semi-elasticity of  $\varphi_2$  with respect to  $\mu$ . Analogous to Pratt's analysis of risk aversion in the small, we have that the invariant symmetric preferences exhibit DAUA (respectively, IAUA) locally if the negative of the semi-elasticity of  $\varphi_1$  with respect to  $\mu$  is less than or equal to (respectively, is greater than or equal to) the semi-elasticity of  $\varphi_2$  with respect to  $\mu$ .

For (additively) separable  $\varphi$ , that is, where  $\varphi_{12} = 0$ , applying inequality (4) yields that DAUA holds if and only if  $\varphi$  is convex in  $\mu$  and IAUA holds if and only if  $\varphi$  is concave in  $\mu$ . By combining these last two implications we have:

**Corollary 6** *Suppose that the preferences  $\succsim$  admit the invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ .*

*Then the following are equivalent: (i) preferences exhibit CAUA; and (ii)  $\varphi(\mu, \rho) = \mu - \phi(\rho)$ , for some increasing function  $\phi(\cdot)$ .*

We can easily derive the implication of decreasing absolute uncertainty premia as defined by  $\varphi(\mu, 0) - \varphi(\mu, \rho)$ .

**Corollary 7** *Suppose that the preferences  $\succsim$  admit the canonical invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$  and exhibit DAUA (respectively, IAUA). Then, for all  $\bar{\rho}$  in  $\rho(U(X)^n)$ ,*

*(a) the absolute uncertainty premium  $[\varphi(\mu, 0) - \varphi(\mu, \bar{\rho})]$  is non-increasing (respectively, non-decreasing) in  $\mu$ ; and*

*(b)  $\varphi(\mu + \delta, \bar{\rho}) \geq$  (resp.  $\leq$ )  $\varphi(\mu, \bar{\rho}) + \delta$ .*

One set of examples of invariant symmetric preferences that allow for varying premia are those that correspond to Epstein's (1985) generalized mean-variance preferences (translated from risk to uncertainty). But those preferences also violate monotonicity. The following is an example of invariant symmetric preferences that exhibit decreasing absolute uncertainty aversion but are monotone.

**Example 1** Consider the mean-dispersion representation  $\langle U, \pi, \rho, \varphi \rangle$  where  $U$  is a bounded affine utility function in which  $U(w) = -1$  and  $U(z) = 1$ ;  $\pi$  is a probability;  $\rho(u) := \sum_s \pi_s |u_s - E_\pi(u)|$ ; and  $\varphi(\mu, \rho) := \mu - \kappa(\mu) \log(1 + \rho)$  where  $\kappa : U(X) \rightarrow [0, 1]$  is a twice-differentiable function with  $\kappa' < 0$ ,  $\kappa'' < 0$ ,  $\kappa(-1) = 1/4$  and  $\kappa(1) = 0$ .

This example may be viewed as a generalization of preferences introduced by Ergin and Gul (2009). Ergin and Gul's preferences have  $\kappa(\mu) \equiv 1/4$ , hence are quasi-linear in  $\mu$ , and so exhibit constant absolute uncertainty aversion. The preferences in this example are only weakly separable since  $\mu$  appears in the term  $\kappa(\mu)$ . It is straightforward to see that these preferences exhibit the property that the absolute uncertainty premium is decreasing in  $\mu$  since the weight  $\kappa(\mu)$  put on  $\log(1 + \rho)$  is decreasing in  $\mu$ .<sup>10</sup> They also exhibit DAUA since

$$\frac{-\varphi_{11}(\mu, \rho)}{\varphi_1(\mu, \rho)} = \frac{\kappa''(\mu) \log(1 + \rho)}{1 - \kappa'(\mu) \log(1 + \rho)} \leq 0 < \frac{-\kappa'(\mu)}{\kappa(\mu)} = \frac{\varphi_{12}(\mu, \rho)}{-\varphi_2(\mu, \rho)},$$

ensures that inequality (4) holds everywhere.

## 7 Choice and comparative statics

Choice problems for individuals with invariant symmetric preferences are especially convenient analytically. First, for a very large class of problems, one can isolate an 'efficient frontier' and then from that efficient frontier pick an optimal dispersion exposure as characterized by  $\rho$ . To illustrate, consider the general choice problem in which  $F$ , the set of acts from which the individual may choose, is a closed and convex subset of  $\mathcal{F}$ . If her preferences admit an invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$  then her choice problem may be expressed as:

$$\max_f \{ \varphi(E_\pi(U \circ f), \rho(U \circ f)) : f \in F \}.$$

Assume that a well defined solution denoted by  $f^*$  exists to this problem and set  $\mu^* := E_\pi(U \circ f^*)$  and  $\rho^* := \rho(U \circ f^*)$ .

Because  $\varphi$  is increasing in  $\mu$  and decreasing in  $\rho$ , this optimization problem can be rewritten

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<sup>10</sup> Since  $\kappa(\mu) \leq 1/4$  for all  $\mu$  in  $U(X)$ , it follows from Ergin & Gul's result that the preferences in this example are monotonic.



as:

$$\max_{\mu} \{\varphi(\mu, \hat{\rho}(\mu, F))\},$$

where

$$\hat{\rho}(\mu, F) = \min_f \{\rho(U \circ f) : f \in F, E_{\pi}(U \circ f) = \mu\}.$$

Here  $\hat{\rho}(\cdot, F)$  characterizes the ‘efficient frontier’ for mean-dispersion trade-offs, where dispersion is measured by  $\rho(u)$ . Because  $\rho(\cdot)$  is sublinear, and  $F$  is closed and convex, this first-stage programming problem is amenable to simple convex programming tools. Furthermore, if  $\varphi$  is suitably smooth and quasi-concave, the optimal choice problem now reduces to equating a generalization of Epstein’s (1985) “generalized Arrow-Pratt risk aversion measure” to the slope of that efficient frontier

$$-\frac{\varphi_2(\mu^*, \rho^*)}{\varphi_1(\mu^*, \rho^*)} = \frac{1}{\hat{\rho}_{\mu}(\mu^*, F)}. \quad (5)$$

Hence,  $1/\hat{\rho}_{\mu}(\mu^*, F)$  may be interpreted as a (marginal) uncertainty premium. This tangency condition characterizing the optimal choice in  $(\rho, \mu)$  space for smooth  $\varphi$  is illustrated in figure 5.

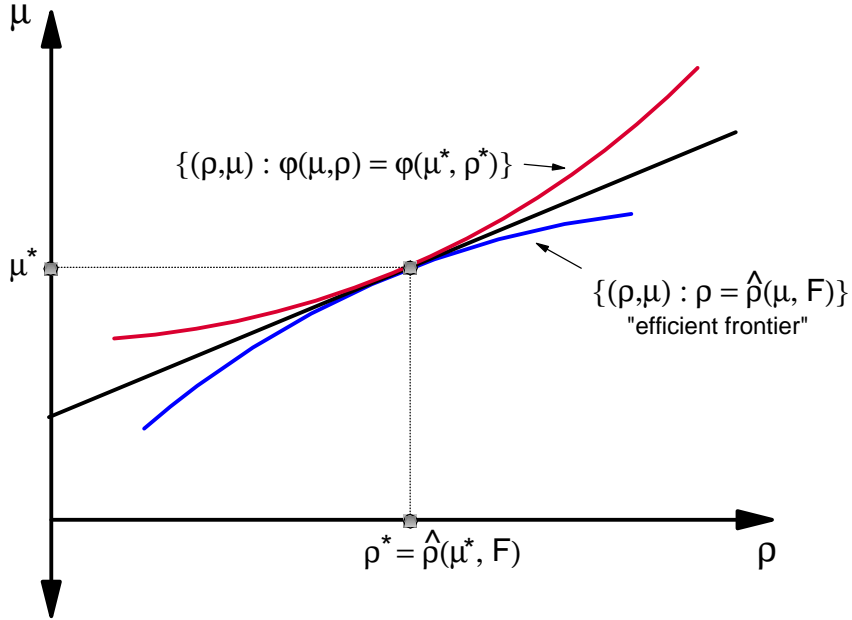


Figure 5: Optimal choice is characterized by  $(\rho^*, \mu^*)$  on the efficient frontier  $\{(\rho, \mu) : \rho = \hat{\rho}(\mu, F)\}$  that is tangent with highest attainable indifference curve of  $\varphi(\cdot, \cdot)$ . That is, the point  $(\rho^*, \mu^*)$  where  $-\varphi_2(\mu^*, \rho^*)/\varphi_1(\mu^*, \rho^*) = [\hat{\rho}_{\mu}(\mu^*, F)]^{-1}$ .

Moreover, when the set of state-utility vectors  $\hat{U} = \{u \in [-1, 1]^n : u = U \circ f, \text{ for some } f \in F\}$  induced by the choice set  $F$  is a cone, then for  $\mu > 0$

$$\begin{aligned}\hat{\rho}(\mu, F) &= \min_u \left\{ \rho(u) : u \in \hat{U}, E_\pi(u) = \mu \right\} \\ &= \min_{\frac{u}{\mu}} \left\{ \mu \rho\left(\frac{u}{\mu}\right) : \frac{u}{\mu} \in \hat{U}, E_\pi\left(\frac{u}{\mu}\right) = 1 \right\} \\ &= \mu \hat{\rho}(1, F).\end{aligned}$$

so that the efficient frontier is linear.

Because preferences may be expressed in terms of two parameters, the usual results for demand theory with two goods are applicable. That is, any comparative static problem involves a substitution effect (which may always be signed unambiguously) and an income effect (which may be signed on the basis of the assumption of DAUA or IAUA). This enables a simple characterization of the optimal solution, and derivation of comparative static results (see, for example, Tobin [1958]; Meyer [1987]; Ormiston and Quiggin [1993]; Ormiston and Schlee [2001]).

In particular, a compensated reduction in the uncertainty premium  $1/\hat{\rho}_\mu(\mu, F)$  must reduce the optimal level of  $\rho$ . Furthermore, given linear utility over wealth, an increase in base wealth (or equivalently, a translation of the set  $\hat{U}$  of induced state-utility vectors in the direction of the  $\mu$ -axis), the optimal level of  $\rho$  will increase if and only if preferences display DAUA. Exposure to additional, non-diversifiable uncertainty is, under weak conditions, equivalent in its effects to a reduction in base wealth. This may be seen by considering that if the initially optimal element of the choice set is adjusted to restore the level of dispersion that prevailed in the absence of the additional risk, the effect will be to reduce mean return. At the new point, with lower mean return and the uncertainty parameter  $\rho$  unchanged, we will have  $-\varphi_2(\mu, \hat{\rho})/\varphi_1(\mu, \hat{\rho}) > 1/\hat{\rho}_\mu(\mu, F)$  and equality can be restored only with a further reduction in both  $\rho$  and  $\mu$ .

We record these points as a proposition, stated for the case of an affine efficient frontier in  $(\rho, \mu)$  space.

**Proposition 8** *Suppose the decision maker's preferences  $\succsim$  on  $\mathcal{F}$  admit an invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$  and display DAUA. Assume the efficient frontier for the choice set  $F$  corresponds to the line in  $(\rho, \mu)$  space  $\hat{\rho} = \rho_{\min} + \hat{\rho}_\mu \max\{\mu - \mu_0, 0\}$  where:*

\*  $\rho_{\min} := \min \{\hat{\rho}(\mu, F)\}$  is undiversifiable background risk;

\*  $\mu_0 = \operatorname{argmin}_{\mu} \hat{\rho}(\mu, F)$ , is the mean utility of the element of  $F$  that minimizes dispersion

\*  $\hat{\rho}_{\mu}$  is a constant.

Let  $\mu^* = \operatorname{argmax}_{\mu} \varphi(\mu, \hat{\rho}(\mu, F))$  denote the mean utility of the optimal act in  $F$ .

Now consider a change in the choice set from  $F$  to  $F'$ , characterized by the change in the affine frontier to  $\hat{\rho}' = \rho'_{\min} + \hat{\rho}'_{\mu} \max\{\mu - \mu'_0, 0\}$

(i) If  $\rho'_{\min} = \rho_{\min}$ ,  $\hat{\rho}'_{\mu} = \hat{\rho}_{\mu}$  and  $\mu'_0 < \mu_0$ , then  $\mu^* - \mu^{*'} \geq \mu_0 - \mu'_0 > 0$ .

(ii) If  $\rho'_{\min} = \rho_{\min}$ ,  $\hat{\rho}'_{\mu} > \hat{\rho}_{\mu}$  and  $\mu'_0 = \mu_0$ , then  $\mu^{*'} < \mu^*$

(iii)  $\rho'_{\min} > \rho_{\min}$ ,  $\hat{\rho}'_{\mu} = \hat{\rho}_{\mu}$  and  $\mu'_0 = \mu_0$ ,  $\hat{\rho}(\mu^{*'}, F) \leq \hat{\rho}(\mu^*, F)$ .

Figure 6 illustrates the result in Proposition 8 (i) that shows how the optimum choice of the mean by a decision-maker who exhibits DAUA falls by more than the vertical parallel downward shift in the linear efficient frontier.

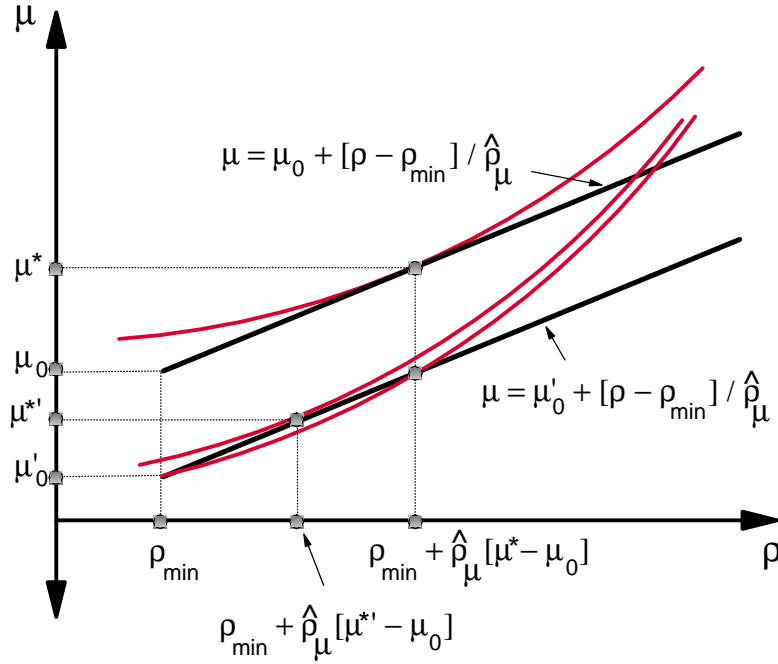


Figure 6: Illustrates how  $(\rho^*, \mu^*)$  changes for  $\varphi(\cdot, \cdot)$  that exhibits DAUA when the linear efficient frontier shifts as a result of the  $\mu$  associated with  $\rho_{\min}$  is reduced from  $\mu_0$  to  $\mu'_0$ .

For general convex choice sets, applying these results to the line tangent to the set of state-

utility vectors induced by the choice set at the initial optimum provides a characterization of local comparative statics.

These results may usefully be compared with the corresponding analysis under expected utility. Analogs to results (i) and (ii) are derived by Sandmo (1971) and subsequent writers. However, the comparative static effects of an increase in background risk in expected utility theory, discussed in most detail by Gollier (2001) cannot in general be decomposed into a two-dimensional summary statistic, and can involve conditions that depend on the fourth or fifth derivatives of the Bernoulli utility of wealth function.

## 7.1 Two-fund separation and an Asset Pricing Formula

These comparative static results have economically significant implications, most notably in the portfolio problem, where a version of two-fund separation applies and which admits a CAPM style pricing formula.

For ease of exposition, we shall assume that  $X$  is the real line, and the decision maker's preferences admit an invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ .<sup>11</sup> Because our interest is in the investor's preferences over trade offs between expected return and the state-contingent dispersion of returns, we take the affine utility  $U$  to be the identity function  $I(x) \equiv x$ , enabling us to identify the choice set  $F$  (the set of feasible state-contingent returns) with  $\hat{U}$ , a set of state-utility vectors. Furthermore, suppose  $\hat{U}$  is generated by the set of portfolios made up from a set of assets, one of which is a safe asset with return vector  $re$ . More precisely, let  $u^j \in \mathbb{R}^n$ ,  $j = 0, 1, \dots, J$  be the return vector on asset  $j$ , and let  $\alpha^j \in \mathbb{R}$  be the holding of asset  $j$ , with (normalized) price equal to 1. Let asset 0 be the safe asset with return  $r$ , so that  $u^0 = re$ . For  $j = 1, \dots, J$ , set  $\bar{r}^j := E_\pi(u^j)$ , that is, the (subjective) expected return of asset  $j$  from the perspective of the investor.

Denote holdings of the non-safe assets by  $\alpha = (\alpha^1, \dots, \alpha^J)$  and the holding of the safe asset

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<sup>11</sup> This is not strictly within the framework of our characterization but we consider the natural extension of our model to this setting of unbounded utility.

by  $\alpha^0$ . The portfolio problem for an investor with initial wealth  $W$  is thus:

$$\begin{aligned} & \max_{\langle \alpha^0, \alpha \rangle} \left\{ \varphi(E_\pi(u), \rho(u)) : \alpha^0 + \sum_{j=1}^J \alpha^j = W, u = \alpha^0 r e + \sum_{j=1}^J \alpha^j u^j \right\} = \\ & \max_{\langle \alpha \rangle} \varphi \left( r \left( W - \sum_{j=1}^J \alpha^j \right) + \sum_{j=1}^J \alpha^j \bar{r}^j, \rho \left( \sum_{j=1}^J \alpha^j u^j \right) \right), \end{aligned}$$

after using the translation invariance of  $\rho$ . Letting  $\bar{r}$  denote the expected return (per dollar invested) of the portfolio, then for this problem

$$\begin{aligned} \hat{\rho}(W\bar{r}, \hat{U}) &= \min_{\langle \alpha \rangle} \left\{ \rho \left( \sum_{j=1}^J \alpha^j u^j \right) : W\bar{r} = \left( W - \sum_{j=1}^J \alpha^j \right) r + \sum_{j=1}^J \alpha^j \bar{r}^j \right\} \\ &= \min_{\langle \alpha \rangle} \left\{ \rho \left( \sum_{j=1}^J \alpha^j u^j \right) : W(\bar{r} - r) = \sum_{j=1}^J \alpha^j (\bar{r}^j - r) \right\} \\ &= \begin{cases} W(\bar{r} - r) \min_{\langle \alpha \rangle} \left\{ \rho \left( \sum_{j=1}^J \alpha^j u^j \right) : 1 = \sum_{j=1}^J \alpha^j (\bar{r}^j - r) \right\} & \text{if } \bar{r} > r \\ 0 & \text{if } \bar{r} = r \end{cases} \\ &= \begin{cases} W(\bar{r} - r) \hat{\rho}(1, \hat{U}_{-0} + \{-re\}) & \text{if } \bar{r} > r \\ 0 & \text{if } \bar{r} = r \end{cases} \end{aligned} \quad (6)$$

where

$$\hat{\rho}(1, \hat{U}_{-0} + \{-re\}) := \min_{\langle \alpha \rangle} \left\{ \rho \left( \sum_{j=1}^J \alpha^j u^j \right) : 1 = \sum_{j=1}^J \alpha^j (\bar{r}^j - r) \right\},$$

and  $\hat{U}_{-0} + \{-re\}$  denotes the set of feasible state contingent ‘excess’ return vectors that can be achieved through the choice of a portfolio of non-safe assets (that is, a portfolio with a zero holding of the safe asset) and where excess return of an asset is the difference between its return and that of the safe asset.

This decomposition shows that if all investors use the same base-line measure  $\pi$  and the same measure of dispersion  $\rho(\cdot)$ , then regardless of their attitudes towards mean and dispersion as encoded in the aggregator  $\varphi$ , any interior solution to the investment problem (that is, where an investor chooses a portfolio with an expected return  $\bar{r} > r$ ) satisfies two-fund separation (the mutual-fund principle). Each investor  $h$  whose preferences admit the invariant symmetric representation  $\langle I, \pi, \rho, \varphi^h \rangle$  chooses some linear combination of the safe asset and the portfolio of

non-safe assets defined by:

$$\hat{\alpha} \in \arg \min \left\{ \rho \left( \sum_{j=1}^J \alpha^j u^j \right) : 1 = \sum_{j=1}^J \alpha^j (\bar{r}^j - r) \right\}. \quad (7)$$

Moreover, applying Proposition 8 (ii) with  $\rho_{\min} = 0$ ,  $\mu_0 = r$ , and  $\hat{\rho}_\mu = \hat{\rho}(1, \hat{U}_{-0} + \{-re\})$  yields:

**Corollary 9** *An investor with invariant symmetric preferences that exhibit DAUA will respond to an increase in  $\hat{\rho}(1, \hat{U}_{-0} - re)$  (the measure of dispersion of the minimum-dispersion unit excess-return portfolio of non-safe assets), by decreasing the mean return of her optimal portfolio.*

If  $\rho(\cdot)$  is ‘smooth enough’ to have a gradient  $\nabla \rho$ ,<sup>12</sup> the first-order conditions for (7) require for each  $j$ , evaluated at the optimal portfolio  $\hat{\alpha}$ :

$$\left\langle \nabla \rho \left( \sum_{k=1}^J \hat{\alpha}^k u^k \right), u^j \right\rangle - \lambda (\bar{r}^j - r) = 0, \quad (8)$$

where  $\lambda$  is the associated Lagrangean multiplier and  $\langle v, v' \rangle$  denotes the inner product of the two vectors  $v$  and  $v'$ . Multiplying both sides of (8) by  $\hat{\alpha}^j$  and summing, we obtain:

$$\begin{aligned} \lambda \sum_{j=1}^J \hat{\alpha}^j (\bar{r}^j - r) &= \left\langle \nabla \rho \left( \sum_{k=1}^J \hat{\alpha}^k u^k \right), \sum_{j=1}^J \hat{\alpha}^j u^j \right\rangle \\ &= \rho \left( \sum_{j=1}^J \hat{\alpha}^j u^j \right), \end{aligned}$$

where the second equality follows by the linear homogeneity of  $\rho$ . Recall from the constraint for (7) that  $\sum_{j=1}^J \hat{\alpha}^j (\bar{r}^j - r) = 1$ , hence substituting for  $\lambda$  in (8) gives for each  $j$ :

$$\bar{r}^j - r = \frac{\left\langle \nabla \rho \left( \sum_{k=1}^J \hat{\alpha}^k u^k \right), u^j \right\rangle}{\rho \left( \sum_{j=1}^J \hat{\alpha}^j u^j \right)}. \quad (9)$$

Let  $\bar{r}^M$  denote the expected return per dollar spent on the efficient mutual fund. By definition,

$$\bar{r}^M = \sum_{j=1}^J \gamma^j \bar{r}^j, \text{ where } \gamma^j = \frac{\hat{\alpha}^j}{\sum_{k=1}^J \hat{\alpha}^k}.$$

Notice that  $\gamma^j$  is the fraction of each dollar spent on the efficient mutual fund that is used to purchase asset  $j$ .

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<sup>12</sup> Since  $\rho(\cdot)$  is convex, the gradient will exist almost everywhere. Where it does not exist there will exist a subdifferential and we can use a one-sided directional derivative instead.

To express (9) in a more familiar form, first notice that:

$$\bar{r}^M - r = \sum_{j=1}^J \gamma^j (\bar{r}^j - r) = \frac{\sum_{j=1}^J \hat{\alpha}^j (\bar{r}^j - r)}{\sum_{k=1}^J \hat{\alpha}^k}.$$

Also from the homogeneity of degree zero of  $\nabla \rho$  and the linear homogeneity of  $\rho$ , we have:

$$\begin{aligned} \nabla \rho \left( \sum_{k=1}^J \hat{\alpha}^k u^k \right) &= \nabla \rho \left( \frac{\sum_{k=1}^J \hat{\alpha}^k u^k}{\sum_{k=1}^J \hat{\alpha}^k} \right) = \nabla \rho \left( \sum_{k=1}^J \gamma^k u^k \right), \\ \text{and } \frac{\rho \left( \sum_{j=1}^J \hat{\alpha}^j u^j \right)}{\sum_{k=1}^J \hat{\alpha}^k} &= \rho \left( \frac{\sum_{j=1}^J \hat{\alpha}^j u^j}{\sum_{k=1}^J \hat{\alpha}^k} \right) = \rho \left( \sum_{j=1}^J \gamma^j u^j \right). \end{aligned}$$

So, if we multiply the right-hand side of (9) by  $\sum_{j=1}^J \hat{\alpha}^j (\bar{r}^j - r)$  ( $= 1$ ), and then multiply and divide it by  $\sum_{k=1}^J \hat{\alpha}^k$ , we obtain:

$$\bar{r}^j - r = \frac{\left\langle \nabla \rho \left( \sum_{k=1}^J \hat{\alpha}^k u^k \right), u^j \right\rangle \sum_{j=1}^J \hat{\alpha}^j (\bar{r}^j - r)}{\rho \left( \sum_{j=1}^J \hat{\alpha}^j u^j \right) / \sum_{k=1}^J \hat{\alpha}^k} = \frac{\left\langle \nabla \rho \left( \sum_{k=1}^J \gamma^k u^k \right), u^j \right\rangle}{\rho \left( \sum_{j=1}^J \gamma^j u^j \right)} (\bar{r}^M - r).$$

Thus for each asset  $j$  :

$$\bar{r}^j = r + \beta^j (\bar{r}^M - r), \text{ where } \beta^j = \frac{\left\langle \nabla \rho \left( \sum_{k=1}^J \gamma^k u^k \right), u^j \right\rangle}{\rho \left( \sum_{j=1}^J \gamma^j u^j \right)}. \quad (10)$$

The interpretation of  $\beta^j$  as the ‘generalized beta’ parallels the standard CAPM model. Each asset’s generalized beta measures the ratio of the increase in dispersion by spending an extra dollar on that asset to the increase in dispersion by spending an extra dollar on the efficient mutual fund  $(\gamma^1, \dots, \gamma^J)$ .<sup>13</sup>

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<sup>13</sup> As a quick check we see that for the case where the measure of dispersion is the standard deviation, that is,  $\rho(u) = \sqrt{\sum_{s=1}^n \pi_s [u_s - E_\pi(u)]^2}$ , the gradient is given by:

$$\nabla \rho(u) = \frac{(\pi_1 [u_1 - E_\pi(u)], \dots, \pi_n [u_n - E_\pi(u)])}{\sqrt{\sum_{s=1}^n \pi_s [u_s - E_\pi(u)]^2}}.$$

Hence,

$$\begin{aligned} \beta^j &= \frac{\sum_{s=1}^n \pi_s \left[ \left( \sum_{k=1}^J \gamma^k u_s^k \right) - E_\pi \left( \sum_{k=1}^J \gamma^k u^k \right) \right] u_s^j}{\sum_{s=1}^n \pi_s \left[ \left( \sum_{k=1}^J \gamma^k u_s^k \right) - E_\pi \left( \sum_{k=1}^J \gamma^k u^k \right) \right]^2} \\ &= \frac{\text{COV} \left[ \left( \sum_{k=1}^J \gamma^k u^k \right), u^j \right]}{\text{VAR} \left[ \sum_{k=1}^J \gamma^k u^k \right]} \end{aligned}$$

which corresponds to the  $\beta^j$  in the standard CAPM formula for asset returns.

## 8 Concluding comments

The simple risk-return analysis of choice under uncertainty was derived under the highly restrictive assumption of mean-variance preferences with known probabilities. Expected utility theory provided a more general framework for analysis, but yielded more limited comparative static results, particularly in relation to background risk.

In this paper, we have shown that the core of this analysis may be extended to encompass more realistic models of preferences, in which individuals are not assumed to be (second-order) probabilistically sophisticated (in the sense of Ergin and Gul [2009]) and in which the strong EU independence axiom is replaced with the weaker requirement of complementary independence, common-mean uncertainty aversion and common-mean certainty independence. The class of models described in this way includes the mean-variance model as a special case, but allows for a wide variety of dispersion measures.

From the viewpoint of economists interested in modelling problems involving uncertainty, rather than in the axiomatic details of decision theory, our message is a positive one. Under fairly general conditions, the standard economic logic of choice, applied to appropriate measures of mean return and more general measures of dispersion, yield analogs of the standard results.

## Appendix: Proofs

### Proof of the Lemma 1.

We show the uniqueness of the canonical invariant symmetric representation below in the proof of our main theorem (theorem 3) immediately following the sufficiency proof that establishes axioms A.1-A.7 imply that the preferences admits a compact invariant symmetric representation.

### Proof of the main theorem (theorem 3).

**(Sufficiency of Axioms)** In the following, we assume A.1 (weak order), A.2 (Continuity), A.3 (monotonicity) and A.4 (best and worst outcome) are given.

We first find an affine utility representation for the preferences restricted to constant acts.



**Lemma 10 (Expected Utility on Constant Acts)** *Given A.5 (Complementary independence), the restriction of preferences to constant acts admits an expected utility representation. That is, there exists a mixture continuous affine utility function  $U : X \rightarrow \mathbb{R}$ , with  $U(w) = -1$ ,  $U(z) = 1$  and such that, for all  $x, y$  in  $X$ ,  $U(x) \geq U(y)$  if and only if  $x \succsim y$ .*

**Proof.** Since any pair of constant acts constitute by definition a pair of complementary acts, consider any three constant acts  $x, x'$  and  $x''$ . If we take  $f = x$ ,  $\bar{f} := x'$  and  $g = \bar{g} = x''$ , and apply the axiom then it follows if  $x \succsim x'$  then  $\alpha x + (1 - \alpha)x'' \succsim \alpha x' + (1 - \alpha)x''$ , for all  $\alpha$  in  $(0, 1)$ . That is, the restriction of  $\succsim$  to  $X$  satisfies the standard independence axiom and along with A.1 to A.4, the hypotheses of the expected utility theorem. As the function  $U$  is unique up to affine transformations, we can without loss of generality set  $U(w) := -1$  and  $U(z) := 1$ .  $\square$

Let  $x_0$  denote a constant act for which  $U(x_0) = 0$ .

Recall the following definition from section 3:

**Definition 11 (Induced Preferences)** *2 Let  $\succsim_u$  be the binary relation on  $[-1, 1]^n$  defined by  $u \succsim_u u'$  if there exists a corresponding pair of acts  $f$  and  $f'$  in  $F$  with  $U \circ f = u$  and  $U \circ f' = u'$ , such that  $f \succsim f'$ .*

**Lemma 11 (State-Utility Preferences)** *Let  $U(\cdot)$  be an affine representation of  $\succsim$  on  $X$  with  $U(X) = [-1, 1]$ . The binary relation  $\succsim_u$  inherits order, continuity and monotonicity. In particular,  $u \succsim_u u'$  if and only if for all acts  $f$  and  $f'$  in  $\mathcal{F}$  such that  $U \circ f = u$  and  $U \circ f' = u'$ , we have  $f \succsim f'$ .*

**Proof.** Completeness follows from the affineness of  $U$ . That is, for any  $u$  in  $[-1, 1]^n$ , there exists an act  $f$  in  $\mathcal{F}$  with  $U \circ f = u$ , for example, the act  $f$  where  $f(s) := z(u_s + 1)/2 + w(-u_s + 1)/2$ . Furthermore, for any pair of acts  $f$  and  $g$ , if  $U \circ f = U \circ g$ , then  $f(s) \sim g(s)$  for all  $s$  in  $S$ , hence by A.3 (monotonicity)  $f \sim g$ . Hence  $U \circ f \succsim_u U \circ g$  if and only if  $f \succsim g$ . Similarly,  $U \circ f \succ U \circ g$  if and only if  $f \succ g$  and  $U \circ f \sim_u U \circ g$  if and only if  $f \sim g$ . Hence transitivity and monotonicity are inherited by  $\succsim_u$ . To establish continuity, fix three utility vectors  $u, u'$  and  $u''$  in  $U \circ \mathcal{F}$  (i.e., in  $[-1, 1]^n$ ) such that  $u' \succsim_u u \succsim u''$ . We need to show that the sets

$\{\alpha \in [0, 1] : \alpha u' + (1 - \alpha) u'' \succsim_u u\}$  and  $\{\alpha \in [0, 1] : u \succsim_u \alpha u' + (1 - \alpha) u''\}$  are closed. Let  $f, f'$  and  $f''$  be such that  $U \circ f = u, U \circ f' = u',$  and  $U \circ f'' = u''$ . We have  $\alpha u' + (1 - \alpha) u'' \succsim_u u$  if and only if  $aU \circ f' + (1 - \alpha)U \circ f'' \succsim_u U \circ f$  if and only if  $U \circ (af' + (1 - \alpha)f'') \succsim_u U \circ f$  if and only if  $af' + (1 - \alpha)f'' \succsim f$ . Hence the set  $\{\alpha \in [0, 1] : \alpha u' + (1 - \alpha) u'' \succsim_u u\}$  is equal to the set  $\{\alpha \in [0, 1] : \alpha f' + (1 - \alpha) f'' \succsim f\}$ , and the set  $\{\alpha \in [0, 1] : u \succsim_u \alpha u' + (1 - \alpha) u''\}$  is equal to the set  $\{\alpha \in [0, 1] : f \succsim_u \alpha f' + (1 - \alpha) f''\}$ , which are closed by A.2.  $\square$

We next show that every act has a certainty equivalent.

**Definition 12** *For all acts  $f$  in  $\mathcal{F}$ , we say that a constant act  $x_f$  in  $X$  is a certainty equivalent of  $f$  if  $x_f \sim f$ .*

**Lemma 12 (Certainty Equivalents)** *Given A.5 (Complementary independence), all acts  $f$  in  $\mathcal{F}$  have certainty equivalents.*

**Proof.** To see this, fix  $f$  and consider  $u^f = U \circ f$ . By monotonicity of  $\succsim_u, e \succsim_u u^f \succsim_u -e$ . By continuity and monotonicity of  $\succsim_u$ , there exists unique  $\beta \in [0, 1]$ , such that  $\beta e + (1 - \beta)(-e) \sim_u u^f$ . Hence, by Lemma 11,  $(2\beta - 1)e \sim_u u^f$  if and only if  $\beta z + (1 - \beta)w \sim f$ . Thus  $\beta z + (1 - \beta)w$  is a certainty equivalent of  $f$ .  $\square$

Turning our attention to complementary pairs of acts, notice that if the pair of acts  $(f, g)$  is a complementary pair then  $\frac{1}{2}U \circ f + \frac{1}{2}U \circ g = ke$  for some  $k \in [-1, 1]$ . Results equivalent to the next two lemmas appear in Siniscalchi (2009) but we provide independent proofs. First we show that the induced preferences satisfy the following property.

**Definition 13 (Additivity for Complementary Pairs)** *For any two pairs of complementary utility vectors  $(u, \bar{u})$  and  $(u', \bar{u}')$  in  $[-1, 1]^n \times [-1, 1]^n$ : if  $u \sim_u \bar{u}$  and  $u' \sim_u \bar{u}'$  then  $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$  for all  $\lambda, \gamma \geq 0$ , such that  $\lambda u + \gamma u'$  and  $\lambda \bar{u} + \gamma \bar{u}'$  are both in  $[-1, 1]^n$ .*

**Lemma 13 (Additivity for complementary pairs)** *The induced preferences  $\succsim_u$  exhibit the property ‘additivity for complementary pairs’ if and only if the underlying preferences  $\succsim$  satisfy, axiom A.5, complementary independence.*

**Proof.**

(Sufficieny) Fix two pairs of complementary utility vectors  $(u, \bar{u})$  &  $(u', \bar{u}')$  in  $[-1, 1]^n \times [-1, 1]^n$  and fix  $\lambda, \gamma \geq 0$  such that  $\lambda u + \gamma u'$  and  $\lambda \bar{u} + \gamma \bar{u}'$  are both in  $[-1, 1]^n$ . Suppose  $u \sim_u \bar{u}$  and  $u' \sim_u \bar{u}'$ . Since the pairs are complementary, there exist  $k, k'$  in  $[-2, 2]$ , such that  $u = k - \bar{u}$  and  $u' = k' - \bar{u}'$ .

So consider the four acts  $f : S \rightarrow \Delta(\{w, z\})$ ,  $\bar{f} : S \rightarrow \Delta(\{w, z\})$ ,  $g : S \rightarrow \Delta(\{w, z\})$  and  $\bar{g} : S \rightarrow \Delta(\{w, z\})$ , satisfying  $U \circ f = (\lambda + \gamma)u$ ,  $U \circ \bar{f} = (\lambda + \gamma)\bar{u}$ ,  $U \circ g = (\lambda + \gamma)u'$  and  $U \circ \bar{g} = (\lambda + \gamma)\bar{u}'$ . Notice the pairs  $(f, \bar{f})$  and  $(g, \bar{g})$  are complementary since,  $\frac{1}{2}f + \frac{1}{2}\bar{f} = \frac{(\lambda+\gamma)k+1}{2}z + \frac{(-[\lambda+\gamma]k)+1}{2}w$  and similarly,  $\frac{1}{2}g + \frac{1}{2}\bar{g} = \frac{([\lambda+\gamma]k')+1}{2}z + \frac{(-[\lambda+\gamma]k')+1}{2}w$ .

Now let  $x_0$  be the constant act for which  $U(x_0) = 0$ .

We first show that  $f \sim \bar{f}$  and  $g \sim \bar{g}$ .

Case (i)  $\lambda + \gamma \geq 1$ . Suppose  $f \sim \bar{f}$  fails, and in particular, wlog suppose  $f \succ \bar{f}$ . By continuity there exists  $\varepsilon > 0$ , and an act  $\hat{f}$ , satisfying  $U \circ \hat{f} = U \circ f - \varepsilon e$  and  $\hat{f} \sim \bar{f}$ . Notice that by construction,  $\hat{f}$  is complementary with  $\bar{f}$ . Since  $(x_0, x_0)$  is trivially a complementary pair with  $x_0 \sim x_0$ , applying A.5 Complementary independence we have

$$\frac{1}{\lambda + \gamma} \hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma} x_0 \sim \frac{1}{\lambda + \gamma} \bar{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma} x_0.$$

But since

$$\begin{aligned} U \circ \left[ \frac{1}{\lambda + \gamma} \hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma} x_0 \right] &= u - \varepsilon e, \\ \text{and } U \circ \left[ \frac{1}{\lambda + \gamma} \bar{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma} x_0 \right] &= \bar{u}, \end{aligned}$$

this implies,  $u - \varepsilon e \sim_u \bar{u} \sim_u u$ , contradicting the monotonicity of  $\succsim_u$ .

Case (ii)  $\lambda + \gamma < 1$ . Let  $(f'', \bar{f}'')$  be the complementary pair of acts for which  $U \circ f'' = u$  and  $U \circ \bar{f}'' = \bar{u}$ . Hence  $f'' \sim \bar{f}''$ . Recall  $(x_0, x_0)$  is trivially a complementary pair with  $x_0 \sim x_0$ , so by applying A.8 complementary independence, we have

$$(\lambda + \gamma) f'' + (1 - \lambda - \gamma) x_0 \sim (\lambda + \gamma) \bar{f}'' + (1 - \lambda - \gamma) x_0$$

But since  $U \circ [(\lambda + \gamma) f'' + (1 - \lambda - \gamma) x_0] = U \circ f = u$  and  $U \circ [(\lambda + \gamma) \bar{f}'' + (1 - \lambda - \gamma) x_0] = U \circ \bar{f} = \bar{u}$ ,  $u \sim_u \bar{u}$  implies  $f \sim \bar{f}$ . Similarly, it follows  $g \sim \bar{g}$ .

Applying complementary independence to  $(f, \bar{f})$  and  $(g, \bar{g})$  for  $\alpha = \lambda/(\lambda + \gamma)$  yields  $\alpha f + (1 - \alpha)g \sim \alpha \bar{f} + (1 - \alpha)\bar{g}$ . And since

$$\begin{aligned} U \circ (\alpha f + (1 - \alpha)g) &= \frac{\lambda}{\lambda + \gamma} U \circ f + \frac{\gamma}{\lambda + \gamma} U \circ g = \lambda u + \gamma u' \\ \text{and } U \circ (\alpha \bar{f} + (1 - \alpha)\bar{g}) &= \frac{\lambda}{\lambda + \gamma} U \circ \bar{f} + \frac{\gamma}{\lambda + \gamma} U \circ \bar{g} = \lambda \bar{u} + \gamma \bar{u}', \end{aligned}$$

we have  $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$ , as required.  $\square$

**(Necessity)** Fix a pair of complementary acts  $(f, \bar{f})$ ,  $(g, \bar{g})$  and  $\alpha$  in  $(0, 1)$ . And set  $u := U \circ f$ ,  $\bar{u} := U \circ \bar{f}$ ,  $u' := U \circ g$  and  $\bar{u}' := U \circ \bar{g}$ . Suppose  $u \succsim_u \bar{u}$  and  $u' \succsim_u \bar{u}'$  then if  $u \sim_u \bar{u}$  and  $u' \sim_u \bar{u}'$  by additivity for complementary pairs  $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$  for  $\lambda = \alpha$  and  $\gamma = (1 - \alpha)$ . Hence for the underlying preferences we have:  $\alpha f + (1 - \alpha)g \sim \alpha \bar{f} + (1 - \alpha)\bar{g}$ , as required.

If either of the preferences are strict, for example, say  $u \succ_u \bar{u}$  (and  $\lambda > 0$ ), then by monotonicity and continuity of  $\succsim$  there exists  $\hat{f}$  and  $\varepsilon > 0$ , such that  $U \circ \hat{f} = u - \varepsilon e \sim_u \bar{u}$ . By construction  $\hat{f}$  is complementary to  $\bar{f}$ , so by additivity for complementary pairs we have  $\lambda(u - \varepsilon e) + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$ . Hence by monotonicity of  $\succsim_u$  it follows,  $\lambda u + \gamma u' \succ_u \lambda \bar{u} + \gamma \bar{u}'$  and thus for the underlying preferences we have:  $\alpha f + (1 - \alpha)g \succ \alpha \bar{f} + (1 - \alpha)\bar{g}$ , as required.  $\square$

With this property in hand, we can now construct the baseline measure  $\pi$  to rank complementary state-utility vectors.

**Lemma 14 (Base-line measure)** *Assume A.5. Then there exists a unique  $\pi \in \Delta(S)$ , such that for all pairs of state-utility vectors  $u$  and  $u'$  in  $[-1, 1]^n$ ,  $u$  and  $u'$  have a common mean  $\mu$  if and only if  $\mu = E_\pi(u) = E_\pi(u')$ .*

**Proof.** To aid in this construction, it is convenient to introduce the following notation. Recall  $e = (1, \dots, 1)$ . Let  $e^s$  denote the unit vector with a 1 in the  $s$ th position and zeros elsewhere. Thus for any vector  $u \in \mathbb{R}^n$  we have  $u = \sum_{s=1}^n u_s e^s$ .

For any  $u$  in  $[-1, 1]^n$ , we shall say any vector  $\bar{u}$  in  $\mathbb{R}^n$  is complementary to  $u$ , if  $\bar{u} = ke - u$ , for some  $k \in \mathbb{R}$ .

We will construct the  $\pi$  state-by-state by finding for each unit vector  $e^s$ , a  $\lambda^s$  in  $(0, 1]$  and a complementary vector to which  $\lambda^s e^s$  is indifferent. These two vectors will have the same mean

(according to the baseline measure) and their ‘difference-from-the-mean’ vectors will be the negative complement of each other. But of course the mean of  $\lambda^s e^s$  according to the baseline measure will be  $\lambda^s \pi_s$  and that is how we can recover  $\pi_s$ .

Fix  $s$  in  $S$ . For each  $\hat{\lambda}$  in  $[0, 1]$ , set  $\bar{U}_{\hat{\lambda}}^s := \{u' \in \mathbb{R}^n : u' = ke - \hat{\lambda}e^s\}$ , that is, the set of vectors that are complementary to  $\hat{\lambda}e^s$ . Since by monotonicity  $\hat{\lambda}e^s \succsim_u -\hat{\lambda}e^s$  the set  $\{u' \in [-1, 1]^n : \hat{\lambda}e^s \succsim_u u'\} \cap \bar{U}_{\hat{\lambda}}^s$  is also non-empty. Note also that  $\bar{U}_0^s$  is the constant utility line so  $\{u' \in [-1, 1]^n : u' \succsim_u 0e^s\} \cap \bar{U}_{\hat{\lambda}}^s$  is non-empty. But it may be the case that for sufficiently large  $\hat{\lambda}$  that  $\{u' \in [-1, 1]^n : u' \succsim_u \hat{\lambda}e^s\} \cap \bar{U}_{\hat{\lambda}}^s$  is empty. So we shall set  $\lambda^s$  equal to the maximum  $\hat{\lambda}$  in  $[0, 1]$  for which  $\{u' \in [-1, 1]^n : u' \succsim_u \hat{\lambda}e^s\} \cap \bar{U}_{\hat{\lambda}}^s$  is non-empty. It follows from the continuity and monotonicity of  $\succsim_u$  that there is a unique  $k^s \in [0, 2]$ , such that  $\lambda^s k^s e - \lambda^s e^s \sim_u \lambda^s e^s$ , so set  $\pi_s := k^s/2$ . By monotonicity of  $\succsim_u$ , it follows that  $\pi_s \in [0, 1]$ . Moreover by axiom A.7 (common-mean certainty independence) that  $\lambda k^s e - \lambda e^s \sim_u \lambda e^s$  for all  $\lambda \in (0, \lambda^s]$ . So set  $\underline{\lambda} := \min_s \lambda^s$ . Thus for each  $s$ :

$$(\underline{\lambda} 2\pi_s e - \underline{\lambda} e^s) \sim_u \underline{\lambda} e^s, \text{ and } \frac{1}{2} (\underline{\lambda} 2\pi_s e - \underline{\lambda} e^s) + \frac{1}{2} \underline{\lambda} e^s = \underline{\lambda} \pi_s e.$$

To check that  $E_{\pi}(e) = 1$ , we can apply the ‘additivity for complementary pairs’  $n - 1$  times, by adding the pairs  $\underline{\lambda} 2\pi_s e - \underline{\lambda} e^s$  to obtain:

$$\begin{aligned} & \sum_{s=1}^n (\underline{\lambda} 2\pi_s e - \underline{\lambda} e^s) \sim_u \sum_{s=1}^n \underline{\lambda} e^s \\ \Rightarrow & \underline{\lambda} \left[ \sum_{s=1}^n (2\pi_s - 1) \right] e \sim_u \underline{\lambda} e \end{aligned}$$

So by monotonicity it follows  $2 \sum_{s=1}^n \pi_s - 1 = 1 \Rightarrow \sum_{s=1}^n \pi_s = 1$ , as required.

To obtain the mean of any vector  $u$  in  $[-1, 1]^n$ , let  $k^u \in [-2, 2]$  be the unique number for which  $\lambda k^u e - \lambda u \sim_u \lambda u$ . Now,

$$\lambda k^u e - \lambda u = \sum_{s=1}^n \lambda (k^u - u_s) e^s \sim_u \sum_{s=1}^n \lambda u_s e^s = \lambda u.$$

Recall for each  $s$  we established  $(\underline{\lambda} 2\pi_s e - \underline{\lambda} e^s, \underline{\lambda} e^s)$  is a complementary pair and the two state-utility vectors are indifferent to each other. Applying additivity for complementary pairs  $n - 1$

times we obtain:

$$\begin{aligned} s \quad & \sum_{s=1}^n u_s \underline{\lambda} (2\pi_s e - e^s) \sim_u \sum_{s=1}^n u_s \underline{\lambda} e^s \\ \Rightarrow & 2\mu \underline{\lambda} e - \underline{\lambda} u \sim_u \underline{\lambda} u, \end{aligned}$$

where  $\mu = E_\pi(u)$ . That is, we have shown that  $\mu = k^u/2$ .

Now suppose  $(u, \bar{u})$  ( $u', \bar{u}'$ ) are two complementary pairs of state-utility vectors satisfying  $\underline{\lambda} u \sim \underline{\lambda} \bar{u}$  and  $\underline{\lambda} u' \sim \underline{\lambda} \bar{u}'$ . From the argument above it follows that  $\bar{u} = 2(E_\pi(u))e - u$  and  $\bar{u}' = 2(E_\pi(u'))e - u'$ . From the definition of common mean,  $\underline{\lambda} u$  and  $\underline{\lambda} u'$  share a common mean only if

$$\frac{1}{2} \underline{\lambda} u + \frac{1}{2} \underline{\lambda} \bar{u} = \frac{1}{2} \underline{\lambda} u' + \frac{1}{2} \underline{\lambda} \bar{u}'.$$

The LHS is equal to

$$\underline{\lambda} \left[ \frac{1}{2} u + \frac{1}{2} (2(E_\pi(u))e - u) \right] = \underline{\lambda} (E_\pi(u))e,$$

and the RHS is equal to

$$\underline{\lambda} \left[ \frac{1}{2} u' + \frac{1}{2} (2(E_\pi(u'))e - u') \right] = \underline{\lambda} (E_\pi(u'))e.$$

Hence equating coefficients,  $E_\pi(u) = E_\pi(u')$ , as required.  $\square$

The next three lemmas show that  $\succsim_u$  satisfies convexity, radial homotheticity and translation invariance on common-mean sets, respectively. For all three lemmas let  $U(\cdot)$  be an affine representation of  $\succsim$  on  $X$  for which  $U(w) = -1$  and  $U(z) = 1$ . Let  $\pi$  be the unique base-measure from Lemma 14 For a fixed  $\mu$  in  $[-1, 1]$ , suppose that  $x^\mu$  is a mean for all acts  $f$  that such that  $E_\pi(U \circ f) = \mu$ .

**Lemma 15 (Common-Mean Convexity)** *Assume A.5 (complementary independence) and A.6 (common-mean uncertainty aversion) apply. The restriction of  $\succsim_u$  to state-utility vectors  $u$  in  $[-1, 1]^n$  that such that  $E_\pi(u) = \mu$  is convex.*

**Proof.** Fix  $u$  and  $u'$  in  $[-1, 1]^n$  such that  $E_\pi(u) = E_\pi(u') = \mu$  and suppose wlog  $u \succsim_u u'$ . We need to show  $\alpha u + (1 - \alpha) u' \succsim_u u'$ , for all  $\alpha$  in  $(0, 1)$ . Since  $u \succsim_u u'$  it follows from Lemma 11 that there exist two acts  $f : S \rightarrow \Delta(\{w, z\})$  and  $g : S \rightarrow \Delta(\{w, z\})$  with  $U \circ f = u$ ,  $U \circ g = u'$  and  $f \succsim g$ . Since  $f$  and  $g$  share a common-mean we can applying A.6 (common-mean uncertainty

aversion) to obtain  $\alpha f + (1 - \alpha)g \succsim f$ , hence  $U \circ (\alpha f + (1 - \alpha)g) = \alpha u + (1 - \alpha)u' \succsim_u u'$ , as required.  $\square$

**Lemma 16 (Common-Mean Radial Homotheticity)** *Assume A.5 (complementary independence) and A.7 (certainty invariance) apply. Then for all  $u'$  and  $u''$  in  $[-1, 1]^n$ , such that  $E_\pi(u') = E_\pi(u'') = \mu$  and all  $\alpha \in (0, 1)$ ,  $u' \succsim_u u''$  if and only if  $\alpha u' + (1 - \alpha)\mu e \succsim_u \alpha u'' + (1 - \alpha)\mu e$ .*

**Proof.** By definition  $E_\pi(u) = E_\pi(u') = \mu$  and  $u \succsim_u u'$  if and only if there exists a pair of acts  $f$  and  $g$ , such that  $U \circ f = u$ ,  $U \circ g = u'$  and  $f \succsim g$ . Fix an  $\alpha$  in  $(0, 1)$  and set  $x^\mu := \frac{(\mu+1)}{2}[z] + \frac{(-\mu+1)}{2}[w]$ . Notice that  $U(x^\mu) = \mu$  and so  $x^\mu$  is a common mean for  $f$  and  $g$ . Applying A.7 (certainty invariance) we have  $f \succsim g$  if and only if  $\alpha f + (1 - \alpha)x^\mu \succsim \alpha g + (1 - \alpha)x^\mu$ . And since  $U \circ (\alpha f + (1 - \alpha)x^\mu) = \alpha u' + (1 - \alpha)\mu e$  and  $U \circ (\alpha g + (1 - \alpha)x^\mu) = \alpha u'' + (1 - \alpha)\mu e$ , we have  $u \succsim_u u'$  if and only if  $\alpha u' + (1 - \alpha)\mu e \succsim_u \alpha u'' + (1 - \alpha)\mu e$ , as required.  $\square$

**Lemma 17 (Common-Mean Translation Invariance)** *Assume A.5, and A.7 apply. Then for all  $u$  and  $u'$  in  $[-1, 1]^n$ , such that  $E_\pi(u) = E_\pi(u') = \mu$  and all  $\delta \in \mathbb{R}$  such that  $u + \delta e$  and  $u' + \delta e$  are both in  $[-1, 1]^n$ ,  $u \succsim_u u'$  if and only if  $u + \delta e \succsim_u u' + \delta e$ .*

**Proof.** Fix  $u$  and  $u'$  in  $[-1, 1]^n$ , such that  $E_\pi(u) = E_\pi(u') = \mu$  and without loss of generality consider  $\delta > 0$  such that  $u + \lambda e$  and  $u' + \delta e$  are both in  $[-1, 1]^n$ . Consider the vectors  $v = \lambda^{-1}(u + e) - e$ , and  $v' = \lambda^{-1}(u' + e) - e$ , where  $\lambda = (2 - \delta)/2 (< 1)$ . Notice that

$$\begin{aligned} \lambda v + (1 - \lambda)(-e) &= u + e - \lambda e - (1 - \lambda)e = u, \\ \lambda v + (1 - \lambda)(e) &= u + e - \lambda e + (1 - \lambda)e \\ &= u + 2(1 - \lambda)e = u + \delta e, \\ \lambda v' + (1 - \lambda)(-e) &= u', \\ \lambda v' + (1 - \lambda)(e) &= u' + \delta e. \end{aligned}$$

Since  $v$  lies on the ray from  $(-e)$  that goes through  $u$  as well as lying on the ray from  $(e)$  that goes through  $u + \delta e$ , it follows that  $v$  is in  $[-1, 1]^n$ . Similarly for  $v'$ . Thus there exists a pair of acts  $f : S \rightarrow \Delta(\{w, z\})$  and  $g : S \rightarrow \Delta(\{w, z\})$ , such that  $U \circ f = v$ ,  $U \circ g = v'$ , and from Lemma

11  $v \succsim_u v'$  if and only if  $f \succsim g$ . As  $E_\pi(v) = E_\pi(v') = \lambda^{-1}(\mu + 1) - 1$ , it follows that  $f$  and  $g$  have a common mean, namely, the constant act:

$$\left(\frac{\mu + 1}{2\lambda}\right)[z] + \left(1 - \frac{\mu + 1}{2\lambda}\right)[w],$$

which has utility  $\lambda^{-1}(\mu + 1) - 1$ . Thus applying certainty invariance twice we have  $f \succsim g$  iff  $\alpha f + (1 - \alpha)w \succsim \alpha g + (1 - \alpha)w$  iff  $\alpha f + (1 - \alpha)z \succsim \alpha g + (1 - \alpha)z$ . But by construction:

$$\begin{aligned} U(\lambda f + (1 - \lambda)w) &= \lambda v + (1 - \lambda)(-e) = u, \\ U(\lambda g + (1 - \lambda)w) &= \lambda v' + (1 - \lambda)(-e) = u', \\ U(\lambda f + (1 - \lambda)w) &= \lambda v + (1 - \lambda)e = u + \delta e, \\ U(\lambda g + (1 - \lambda)w) &= \lambda v' + (1 - \lambda)e = u' + \delta e. \end{aligned}$$

Hence  $u \succsim_u u'$  if and only if  $u + \delta e \succsim_u u' + \delta e$ , as required.  $\square$

### Construction of representation.

Let  $U(\cdot)$  be an affine representation of  $\succsim$  on  $X$  for which  $U(w) = -1$  and  $U(z) = 1$ . By Lemma 12, we know for all acts  $f$  in  $\mathcal{F}$ , there exists a constant act  $x(f) \sim f$ . Set  $V(f) := U(x(f))$  to be the representation of  $\succsim$  on  $\mathcal{F}$ . The corresponding representation for the induced preferences  $\succsim_u$  over state utilities  $u'$  in  $[-1, 1]^n$ , is  $W(u') := U(x(f))$  for all  $f$  such that  $U \circ f = u'$ . By Lemma 11, it is enough to show that we can write this representation in the form  $W(u') = \varphi(E_\pi(u'), \rho(u'))$  where  $\pi$  is the unique base-measure from Lemma 14 and where the functions  $\varphi$  and  $\rho$  have the properties stated in Theorem 3.

Let  $\pi$  be the base measure of lemma 14. For  $\mu \in \mathbb{R}$ , let  $H_\pi^\mu$  be the hyperplane  $\{u \in \mathbb{R}^n : E_\pi(u) = \mu\}$ . Suppose the induced preferences  $\succsim_u$  are such that for all  $\mu \in [-1, 1]$ ,  $u \in H_\pi^\mu \cap [-1, 1]^n$  implies  $u \sim \mu e$ . In this case, the preferences are subjective expected utility and have representation  $V(f) = E_\pi(U \circ f)$ , which is the trivial special case of our representation with  $\varphi(\mu, \rho) \equiv \mu$ . So instead assume that there exist a  $\mu \in [-1, 1]$  and a  $\hat{u} \in H_\pi^\mu \cap [-1, 1]^n$  such that  $\mu e \succ_u \hat{u}$ .

For all  $u \in H_\pi^\mu \cap [-1, 1]^n$ , the vector  $\alpha(u - \mu e)$  lies in  $H_\pi^0 \cap [-1, 1]^n$  for  $\alpha$  sufficiently small. By lemma 16, since  $\mu e \succ_u \hat{u}$ , we have  $\mu e \succ_u \alpha \hat{u} + (1 - \alpha)\mu e$  for all  $\alpha \in (0, 1]$ , and by lemma 17,  $\mu e \succ_u \alpha \hat{u} + (1 - \alpha)\mu e$  if and only if  $0 \succ_u \alpha(\hat{u} - \mu e)$  for  $\alpha(\hat{u} - \mu e) \in [-1, 1]^n$ . Hence it is without



loss of generality to take  $\mu = 0$  and  $\hat{u} \in H_\pi^0 \cap [-1, 1]^n$  such that  $0e \succ_u \hat{u}$ .

The preference relation  $\succsim_u$  confined to  $H_\pi^0 \cap [-1, 1]^n$  is homothetic, hence we can extend it to the whole of  $H_\pi^0$  as follows. For all  $u$  in  $H_\pi^0$  let  $\bar{\beta}(u) = \max\{\beta \in (0, 1] : \beta u \in [-1, 1]^n\}$ . Now, with slight abuse of notation, for all  $u, u'$  in  $H_\pi^0$ , let  $u \succsim_u u'$  if and only if  $\beta u \succsim_u \beta u'$  for all  $\beta \in (0, \min\{\bar{\beta}(u), \bar{\beta}(u')\})$ . The extended relation inherits common mean convexity.

Since this preference relation on  $H_\pi^0$  is homothetic with  $0e$  as a bliss point, by Debreu (1960), it admits a linear homogenous representation,  $-\rho$ , with  $\rho(0) = 0$  where  $\rho : H_\pi^0 \rightarrow \mathbb{R}$  is unique up to positive scalar multiplication. To normalize, let  $\rho(\hat{u}) = -W(\hat{u})$ ; that is,  $\rho(\hat{u}) := U(x(f))$  for any  $f$  such that  $U \circ f = \hat{u}$ . Common mean convexity of  $\succsim_u$  and linear homogeneity of  $\rho$  imply that  $\rho$  is subadditive on  $H_\pi^0$ .

We now extend the dispersion measure  $\rho$  from  $H_\pi^0$  to the whole of  $\mathbb{R}^n$  (and in particular to the whole of  $[-1, 1]^n$ ) by setting  $\rho(u) := \rho(u - E_\pi(u)e)$  for any  $u$  in  $\mathbb{R}^n$ . By construction,  $\rho$  is translation invariant, and so the extended function inherits linear homogeneity and subadditivity from its restriction to  $H_\pi^0$ . For all  $u, u'$  in  $[-1, 1]^n$  such that  $E_\pi(u) = E_\pi(u')$ , we have  $\rho(u) \geq \rho(u')$  if and only if  $u' \succsim_u u$ . To see this, fix  $u, u'$  in  $[-1, 1]^n$  such that  $E_\pi(u) = E_\pi(u') = \mu$ , then  $\rho(u) \geq \rho(u')$  if and only if  $u' - \mu e \succsim_u u - \mu e$  (in the extended relation on  $H_\pi^0$ ) if and only if  $\beta(u' - \mu e) \succsim_u \beta(u - \mu e)$  for  $\beta \in (0, \min\{\bar{\beta}(u - \mu e), \bar{\beta}(u' - \mu e)\})$  (in the original relation) if and only if  $\beta(u') \succsim_u \beta(u)$  if and only if  $u' \succsim_u u$ .

Also by construction  $\rho(\mu e) = 0$  for all  $\mu \in [-1, 1]$ , and for all complementary pairs  $(u, \bar{u})$  with common mean (i.e., such that  $\frac{1}{2}u + \frac{1}{2}\bar{u} = E_\pi(u)e = E_\pi(\bar{u})e$ ) we have  $u \sim \bar{u}$  which implies  $\rho(u) = \rho(\bar{u})$ . By the translation invariance of  $\rho$ , this means that for any complementary pair  $(u, \bar{u})$ , with or without a common mean,  $\rho(u) = \rho(\bar{u})$ . In particular,  $\rho(u) = \rho(-\bar{u})$ .

Define  $\varphi(\mu, \rho') := W(u')$  for all  $u'$  in  $[-1, 1]^n$  such that  $E_\pi(u') = \mu$  and  $\rho(u') = \rho'$ . This is well defined since for all  $u', u''$  in  $[-1, 1]^n$  such that  $E_\pi(u') = E_\pi(u'') = \mu$  and  $\rho(u') = \rho(u'')$ , by the definition of  $\rho$ , we have  $u' \sim_u u''$ .

The function is increasing in its first argument and satisfies  $\varphi(\mu, 0) = \mu$  by construction. It is also non-increasing in its second argument since we showed above that for all  $u, u'$  in  $[-1, 1]^n$  such that  $E_\pi(u) = E_\pi(u')$ , we have  $\rho(u) \geq \rho(u')$  if and only if  $u' \succsim_u u$ .  $\square$

**(Unique canonical representation)**

To construct the canonical representation we may first set  $\bar{U}(x) := U(x)$ , since  $U$  obtained above is already appropriately normalized. To find  $\bar{\rho}$ , consider the family of risk measures

$$\{r_\gamma : r_\gamma(u) := \gamma\rho(u) - E_\pi(u), \gamma \geq 0\}.$$

It is immediate that  $r_0(u) \equiv -E_\pi(u)$  is decreasing in  $u$ . But notice that for any  $u$  and  $u'$  in  $[-1, 1]^n$ , for which  $u \geq u'$ ,  $u \neq u'$ , and  $\rho(u) > \rho(u')$ , there exists  $\gamma'(u, u')$ , given by,

$$\gamma'(u, u') = \frac{E_\pi(u) - E_\pi(u')}{\rho(u) - \rho(u')}.$$

From this it follows that for any  $\gamma > \gamma'$ :

$$[r_\gamma(u) - r_\gamma(u')] > \gamma'[\rho(u) - \rho(u')] - [E_\pi(u) - E_\pi(u')] = 0.$$

Let

$$\bar{\gamma} = \inf \{\gamma'(u, u') : u \geq u', u \neq u', \text{ and } \rho(u) > \rho(u')\}.$$

Consider  $\gamma > \bar{\gamma}$ . There exist  $(u, u')$  such that  $u \geq u'$ ,  $u \neq u'$ , and  $\rho(u) > \rho(u')$  such that  $r_\gamma(u) > r_\gamma(u')$ . Hence  $r_\gamma$  is not weakly decreasing.

Thus, we may define the canonical risk measure by setting  $\bar{\rho}(u) := \bar{\gamma}\rho(u)$ , and hence set  $\bar{\varphi}(\mu', \rho') := \varphi(\mu', \rho'/\bar{\gamma})$  to obtain the canonical representation  $\langle \bar{U}, \pi, \bar{\rho}, \bar{\varphi} \rangle$ .  $\square$

**(Necessity of Axioms)** For a mean-dispersion representation  $\langle U, \pi, \rho, \varphi \rangle$  with bounded  $U$ , and with  $\varphi$  and  $\rho$  satisfying

$$[\varphi(E_\pi(u), \rho(u - (E_\pi(u))e)) - \varphi(E_\pi(u'), \rho(u' - (E_\pi(u'))e))] \cdot (u - u') \geq 0$$

for all  $u, u' \in U(X)^n$ , it is immediate that the associated preferences over acts satisfy axioms A.1-A.3.

The next Lemma shows that invariant symmetric preferences satisfy A.6 (common-mean uncertainty aversion).

**Lemma 18 (Common-Mean Uncertainty Aversion)** *Fix a bounded invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ . The associated preferences over acts  $\succsim$  satisfy: for any two acts  $f$  and  $g$  in  $\mathcal{F}$ , if  $f \sim g$  then  $\alpha f + (1 - \alpha)g \succsim f$ .*

**Proof.** Given  $\langle U, \pi, \rho, \varphi \rangle$ ,  $f$  and  $g$  having a common mean and  $f \sim g$  implies  $\varphi(\mu, \rho(u)) = \varphi(\mu, \rho(u'))$ , where  $u = U \circ f$ ,  $u' = U \circ g$  and  $\mu = E_\pi(u) = E_\pi(u')$ . Hence  $\rho(u) = \rho(u')$ ,

$$\begin{aligned} & V(\alpha f + (1 - \alpha)g) \\ &= \varphi(\mu, \rho(\lambda u + (1 - \lambda)u')) \\ &\geq \varphi(\mu, \lambda \rho(u) + (1 - \lambda)\rho(u')) \\ &= \varphi(\mu, \rho(u)) = V(f). \end{aligned}$$

The inequality follows from the convexity of  $\rho$  (which holds since  $\rho$  is both positive linearly homogeneous and sub-additive) and that  $\varphi$  is non-increasing in its second argument.  $\square$

The next lemma shows that invariant symmetric preferences satisfy A.7 (certainty invariance).

**Lemma 19 (Certainty Invariance)** *Fix an invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ . The associated preferences over acts  $\succsim$  satisfy: for any two acts  $f$  and  $g$  in  $\mathcal{F}$ , such that  $E_\pi(U \circ f) = E_\pi(U \circ g)$ , any constant act  $x$  and any  $\alpha$  in  $(0, 1)$ ,  $f \succsim g \Leftrightarrow \alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x$ .*

**Proof.** Given  $\langle U, \pi, \rho, \varphi \rangle$ , it is immediate that the induced preferences  $\succsim_u$  over state-utility vectors satisfy common-mean translation invariance: in fact, for any pair of utility vectors  $u$  and  $u'$  in  $\mathbb{R}^n$ , s.t  $E_\pi(u) = E_\pi(u')$  and any  $\delta \in \mathbb{R}$ ,  $u \succsim_u u' \Rightarrow u + \delta e \succsim_u u' + \delta e$ . Fix acts  $f$  and  $g$  in  $\mathcal{F}$ , such that  $E_\pi(U \circ f) = E_\pi(U \circ g)$  and constant acts  $x$  and  $y$  in  $X$  and  $\alpha$  in  $(0, 1)$ . Set  $\delta := (1 - \alpha)(U \circ y(s) - U \circ x(s))$  and notice that

$$\begin{aligned} & [\alpha U \circ f + (1 - \alpha)U \circ y] - [\alpha U \circ f + (1 - \alpha)U \circ x] \\ &= [\alpha U \circ g + (1 - \alpha)U \circ y] - [\alpha U \circ g + (1 - \alpha)U \circ x] \\ &= \delta e. \end{aligned}$$

Hence  $\alpha U \circ f + (1 - \alpha)U \circ x \succsim_u \alpha U \circ g + (1 - \alpha)U \circ x$  implies  $\alpha U \circ f + (1 - \alpha)U \circ y \succsim_u \alpha U \circ g + (1 - \alpha)U \circ y$ . Therefore  $\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x$  implies  $\alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y$ , as required.  $\square$

Our final lemma show that invariant symmetric preferences exhibit additivity for complementary pairs.

**Lemma 20 (Additivity for complementary pairs II)** *Fix a invariant symmetric representation  $\langle U, \pi, \rho, \varphi \rangle$ . The associated preferences over state-utilities  $\succsim_u$  exhibits the property of ‘additivity for complementary pairs.’*

**Proof.** Take any two pairs of complementary state-utility vectors  $(u, \bar{u})$  and  $(u', \bar{u}')$  and any  $\lambda, \gamma \geq 0$ . If  $u \sim_u \bar{u}$  and  $u' \sim_u \bar{u}'$  then we know from Lemma 14 that  $\frac{1}{2}u + \frac{1}{2}\bar{u} = \mu e$  and  $\frac{1}{2}u' + \frac{1}{2}\bar{u}' = \mu' e$ , where  $\mu = E_\pi(u)$  and  $\mu' = E_\pi(u')$ . Hence,  $E_\pi(\bar{u}) = \mu$ ,  $E_\pi(\bar{u}') = \mu'$ .

Furthermore,  $E_\pi(\lambda u + \gamma u') = E_\pi(\lambda \bar{u} + \gamma \bar{u}') = \lambda \mu + \gamma \mu'$ , and

$$\frac{1}{2}(\lambda u + \gamma u') + \frac{1}{2}(\lambda \bar{u} + \gamma \bar{u}') = (\lambda \mu + \gamma \mu') e$$

So it follows from complementary-symmetry of  $\rho$ , that  $\rho(\lambda u + \gamma u') = \rho(\lambda \bar{u} + \gamma \bar{u}')$ .

$$\begin{aligned} W(\lambda \bar{u} + \gamma \bar{u}') &= \varphi(\lambda \bar{\mu} + \gamma \bar{\mu}', \rho(\lambda \bar{u} + \gamma \bar{u}')) \\ &= \varphi(\lambda \mu + \gamma \mu', \rho(\lambda u + \gamma u')) \\ &= W(\lambda u + \gamma u'), \end{aligned}$$

as required. □

Finally, Lemma 13 demonstrates the necessity of Axiom A.5 (complementary independence) for  $\succsim_u$  to exhibit the property of additivity for complementary pairs. This completes the proof of necessity. □ ■

**Proof of Proposition 4.** Fix acts  $f$  and  $g$  in  $\mathcal{F}$ , a constant act  $y$  in  $X$  and an  $\alpha$  in  $(0, 1)$ . Suppose  $f \succeq g$ . That is, there exists a constant act  $x$ , and a  $\lambda$  in  $[0, 1]$ , such that  $\lambda f + (1 - \lambda)x = g$ . We have

$$\begin{aligned} &\lambda(\alpha f + (1 - \alpha)y) + (1 - \lambda)(\alpha x + (1 - \alpha)y) \\ &= \alpha(\lambda f + (1 - \lambda)x) + (1 - \alpha)y \\ &= \alpha g + (1 - \alpha)y. \end{aligned}$$

Hence,  $\alpha f + (1 - \alpha)y \succeq \alpha g + (1 - \alpha)y$ .

Alternatively, suppose  $\alpha f + (1 - \alpha)y \succeq \alpha g + (1 - \alpha)y$ . By the definition of  $\succeq$ , this entails

that there exists a constant act  $x$  and a  $\lambda$  in  $[0, 1]$ , such that

$$\lambda(\alpha f + (1 - \alpha)y) + (1 - \lambda)(\alpha x + (1 - \alpha)y) = \alpha g + (1 - \alpha)y.$$

Thus,

$$\alpha(\lambda f + (1 - \lambda)x) + (1 - \alpha)y = \alpha g + (1 - \alpha)y.$$

Equating parts of each side of this equation that is multiplied by  $\alpha$ , yields,  $\lambda f + (1 - \lambda)x = g$ .

That is,  $f \succeq g$ , as required.  $\blacksquare$

**Proof of proposition 5 (Decreasing Absolute Uncertainty Aversion).**

Part I. We first establish the following two preliminary results.

I.1.  $f \succeq g \Rightarrow \rho(U \circ f) \geq \rho(U \circ g)$ .

I.2. For any  $\hat{\rho} \in \rho([-1, 1]^n)$ , and any  $0 \leq \rho' < \hat{\rho}$ , there exists acts  $f$  and  $g$  for which  $\rho(U \circ f) = \hat{\rho}$ ,  $\rho(U \circ g) = \rho'$  and  $f \succeq g$ .

Proof of I.1. As  $f \succeq g$ , there exists a constant act  $x$  and  $\lambda \in [0, 1]$ , such that  $g = \lambda f + (1 - \lambda)x$ . Hence  $\rho(U \circ g) = \rho(U \circ [\lambda f + (1 - \lambda)x]) = \alpha \rho(U \circ f) \leq \rho(U \circ f)$  (by the translation invariance and homogeneity of  $\rho(\cdot)$ ).  $\square$

Proof of I.2. To see this, fix a  $\hat{\rho}$  for which  $\rho(\hat{u}) = \hat{\rho}$ , for some  $\hat{u}$  in  $[-1, 1]^n$ . The act  $f$  in which  $f_s = \left(\frac{1+\hat{u}_s}{2}\right)[b] + \left(\frac{1-\hat{u}_s}{2}\right)[w]$ , is an act that by construction is one in which  $U \circ f = \hat{u}$  and hence its measure of dispersion  $\rho(U \circ f) = \rho(\hat{u}) = \hat{\rho}$ . Furthermore,  $\bar{f}$ , where  $\bar{f}_s = \left(\frac{1-\hat{u}_s}{2}\right)[b] + \left(\frac{1+\hat{u}_s}{2}\right)[w]$  is by construction complementary to  $f$ , since  $\frac{1}{2}f + \frac{1}{2}\bar{f} = \frac{1}{2}[b] + \frac{1}{2}[w]$ , a constant act. Finally, for the act

$$g = \frac{\rho'}{\hat{\rho}}f + \left(1 - \frac{\rho'}{\hat{\rho}}\right)\left(\frac{1}{2}[b] + \frac{1}{2}[w]\right),$$

the measure of dispersion of its associated state-utility vector  $\rho(U \circ g) = \rho\left(\frac{\rho'}{\hat{\rho}}\hat{u}\right) = \frac{\rho'}{\hat{\rho}} \times \rho(\hat{u}) = \rho'$  (by the homogeneity of  $\rho(\cdot)$ ), and  $f \succeq g$ , as required.  $\square$

Part II. Proof that (a) implies (b).

In terms of the induced preferences  $\succsim_u$  over state-utility vectors, DAUA requires that if  $\hat{u} \succeq \hat{v}$  then for any  $d < d'$  and any  $\alpha$ ,

$$\alpha\hat{u} + (1 - \alpha)de \sim_u \alpha\hat{v} + (1 - \alpha)de \Rightarrow \alpha\hat{u} + (1 - \alpha)d'e \succsim_u \alpha\hat{v} + (1 - \alpha)d'e$$

or setting  $u := \alpha \hat{u} + (1 - \alpha) de$ ,  $v := \alpha \hat{v} + (1 - \alpha) de$ , and  $\delta := (1 - \alpha)(d' - d) > 0$

$$u \sim_u v \Rightarrow u + \delta e \succsim_u v + \delta e. \quad (11)$$

Notice from the translation invariance of  $\rho(\cdot)$ ,  $\rho(u + \delta e) = \rho(u)$  and  $\rho(v + \delta e) = \rho(v)$ . By proposition 4 it follows that  $u \succeq v$ , so the implication (11) holds whenever the associated vectors are in  $[-1, 1]$ , that is, are in the domain of  $\succsim_u$ .

Now suppose  $\varphi(\mu, \rho) = \varphi(\mu', \rho')$  with  $\mu > \mu'$ . Since  $\varphi$  is increasing in its first argument and non-increasing in its second, it follows that  $\rho > \rho'$ . And from result I.2, we have established that there exists a  $u$  in  $[-1, 1]^n$ , such that  $E_\pi(u) = \mu$  and  $\rho(u) = \rho$ . Since  $\mu' < \mu$  and  $\rho' < \rho$ , it follows there exists a  $u'$  in  $[-1, 1]^n$  such that  $E_\pi(u') = \mu'$  and  $\rho(u') = \rho'$ . Applying (11) yields  $\varphi(\mu + \delta, \rho) \geq \varphi(\mu' + \delta, \rho')$ , as required.  $\square$

Part III. Proof that (b) implies (a).

Fix a pair of acts  $f, g$  in  $\mathcal{F}$  such that  $f \succeq g$  and a pair of constant acts  $y \succsim x$ , and an  $\alpha$  in  $(0, 1)$ . Set  $u := \alpha U \circ f + (1 - \alpha)U(x)e$ ,  $v := \alpha U \circ g + (1 - \alpha)U(x)e$ ,  $\delta := (1 - \alpha)(U(y) - U(x))$ .

Suppose  $\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x$ , that is,  $\varphi(E_\pi(u), \rho(u)) \geq \varphi(E_\pi(v), \rho(v))$ .

By Proposition 4,  $\alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x$ , and applying result I.1 we have  $\rho(\hat{u}) > \rho(v)$ , and hence  $E_\pi(\hat{u}) > E_\pi(v)$ . Now from the continuity and monotonicity of  $\varphi$  wrt its first argument, there exists  $k \geq 0$ , such that for  $\hat{u} := u - ke$ ,  $\varphi(E_\pi(\hat{u}), \rho(\hat{u})) = \varphi(E_\pi(v), \rho(v))$ .

Applying condition (b), in conjunction with the monotonicity of  $\varphi$  wrt its first argument and the translation invariance of  $\rho(\cdot)$ , yields:

$$\varphi(E_\pi(u + \delta e), \rho(u + \delta e)) \geq \varphi(E_\pi(\hat{u}) + \delta, \rho(\hat{u})) \geq \varphi(E_\pi(v) + \delta, \rho(v)) = \varphi(E_\pi(v + \delta e), \rho(v + \delta e)).$$

And

$$\begin{aligned} \varphi(E_\pi(u + \delta e), \rho(u + \delta e)) &\geq \varphi(E_\pi(v + \delta e), \rho(v + \delta e)) \\ &\Rightarrow u + \delta e \succsim_u v + \delta e. \end{aligned}$$

But by construction:  $u + \delta e = \alpha U \circ f + (1 - \alpha)U(y)e$  and  $v + \delta e = \alpha U \circ g + (1 - \alpha)U(y)e$ .

Hence,

$$\alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y,$$

as required. □

Part IV. To show the implication for smooth preferences, consider a path along the indifference curve from  $(\mu, \rho)$  to  $(\mu', \rho')$ , with slope  $\partial\rho/\partial\mu$  given at any point by  $-\varphi_1/\varphi_2$ . Condition (b) holds if and only if  $\varphi_1$  is non-decreasing along this path. The change in  $\varphi_1$  for a small movement along the path is given by  $\varphi_{11} + \varphi_{12}\frac{\partial\rho}{\partial\mu}$ . Hence we require,

$$\begin{aligned} \varphi_{11} + \frac{\varphi_1}{-\varphi_2}\varphi_{12} &\geq 0 \\ \Rightarrow \frac{\varphi_{12}}{-\varphi_2} &\geq \frac{-\varphi_{11}}{\varphi_1}, \end{aligned}$$

as desired. □ ■

### Proof of Corollary 6

(i)  $\Leftrightarrow$  (iii) From **proposition 5 (b)** CAUA holds iff  $\varphi(\mu + \delta, \bar{\rho}) = \varphi(\mu, \bar{\rho}) + \delta$  for all  $\mu, \delta, \rho$  satisfying the stated conditions which is true iff (ii) holds. ■

**Proof of Corollary 7.** (a) is a special case of Proposition 8 part (b), and (b) follows immediately from (a). ■

### Proof of Proposition 8

(i) Consider  $\tilde{\mu} > \mu^*$ . By the optimality of  $\mu^*$ ,  $\varphi(\tilde{\mu}, \hat{\rho}(\tilde{\mu}, F)) \leq \varphi(\mu^*, \hat{\rho}(\mu^*, F))$ . Hence, by DAUA,

$$\begin{aligned} &\varphi(\tilde{\mu} - (\mu_0 - \mu'_0), \rho_{\min} + \hat{\rho}_\mu[\tilde{\mu} - (\mu_0 - \mu'_0)]) \\ &\leq \varphi(\mu^* - (\mu_0 - \mu'_0), \rho_{\min} + \hat{\rho}_\mu[\tilde{\mu} - (\mu_0 - \mu'_0)]). \end{aligned}$$

So the pair  $(\tilde{\mu} - (\mu_0 - \mu'_0), \rho_{\min} + \hat{\rho}_\mu[\tilde{\mu} - (\mu_0 - \mu'_0)])$  cannot be optimal in  $F'$  and hence  $\mu^{*'} \leq \mu^* - (\mu_0 - \mu'_0)$ .

The proofs of parts (ii) and (iii) are similar. ■

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