TITLE: A Matter of Interpretation: Ambiguous Contracts and Liquidated Damages

Authors:

Simon Grant, Jeff Kline & John Quiggin

Working Paper: R11_4

2011

RSMG Working Paper Series

Schools of Economics and Political Science

The University of Queensland

St Lucia

Brisbane

Australia 4072

Web: www.uq.edu.au

20 September 2011

A matter of interpretation: ambiguous contracts and liquidated damages*

Abstract

We focus on syntactic aspects of differential awareness that give rise to contractual disputes. Boundedly rational parties use a common language, but do not share a common understanding of the world, leading to ambiguity in both syntactic and semantic forms. In contractual relationships, ambiguity leads to disagreement and disputes. We show that the agents may prefer simpler less ambiguous contracts when facing potential disputes. In particular, parties may prefer liquidated damages provisions to contractual terms that specify a more complex risk allocation.

JEL Classification: D80, D82

Key words: ambiguity, bounded rationality, expected uncertain utility, incomplete contracts, liduidated damages

Simon Grant
Rice University and University of Queensland

Jeff Kline
University of Queensland (Corresponding Author j.kline@uq.edu.au, +61 7 336 56470)

John Quiggin
School of Economics, University of Queensland, Brisbane 4072, Australia

*We thank Bob Brito, Roger Myerson, Ben Polak, Larry Sameulson, Max Stinchcombe, Matthew Ryan, Nancy Wallace, participants at the 2nd Asian Decentralization Conference, at the 5th Pan-Pacific Conference on Game Theory, at ESAM2009, at LGS6 and at UT's Murray S. Johnson Conference on Bounded Rationality in Economic Theory for helpful comments and criticism.

1 Introduction

Language is a matter of interpretation, and interpretations will differ. This fact is of fundamental importance in the construction of contracts, which are written or verbal agreements that the parties act in particular ways under particular conditions. For any contract to be successfully implemented, the parties must agree on whether the relevant conditions apply. A contract that is ambiguous, in the sense that parties may differ in their interpretation of the conditions that apply (and therefore of the actions that are required, will lead to disputes) and, ultimately, litigation.

To avoid disputes, parties to a contract may seek to avoid ambiguous terms, even when the resulting contract is incomplete, in the sense that opportunities for risk-sharing or productive cooperation are foregone. For example, parties may adopt a standard contract, in which the terms are well-defined as a result of established precedents, even if a variation on the standard contract could potentially yield a Pareto improvement.

The central point that ambiguous contractual terms can lead to incomplete but nevertheless *ex ante* efficient contracts has received relatively little attention from economists.[1] This is because contracts are typically modelled as state-contingent acts, with incompleteness arising from the fact that some states may be non-contractible or from state-contingent preferences that are ambiguous, in the technical sense that there exists no well-defined probability distribution over the state space. The language in which contracts are written is either not specified or is derived from the state space.

In this paper, we begin with a syntactic approach, in which the set of contingencies and the set of actions expressible in a common language available to the two parties are taken as primitive. In this approach, a contract is a set of conditional actions, built up using contingencies that can be expressed using the contractual terms available in the common language. We consider contracts between two parties using the same contractual language, but with possibly different interpretations of the contingencies specified in the contract. We define possibility of dispute relations that specify the pairs of contingencies over which the two players might be in dispute.

[1] Exceptions include Mukerji (1998), Mukerji & Tallon (2001), Board & Chung (2007, 2009), Filiz-Ozbay (2010) and Grant, Kline & Quiggin (2011).

It is natural for a party to consider the range of outcomes that might arise given the ambiguity he or she perceives to be associated with the range of possible interpretations by the other party. We show how this can give rise to preferences that may be represented by the ε-contamination model commonly used to represent preferences averse to state-contingent ambiguity. Thus, our approach establishes a connection between aversion to syntactic or linguistic ambiguity (the sense in which the term 'ambiguity' is normally found in ordinary usage) and semantic or state-contingent ambiguity (the sense in which the term is commonly used in decision theory).

In Section 4, we consider the contractual specification of damages to apply when one party is unable (or finds it undesirable) to fulfil their contractual obligations. In particular, we consider 'liquidated damages' contracts which specify a constant payment for the case of default. We are able to show in Proposition 4 that liquidated damages contracts are *ex ante* efficient when the aversion to ambiguity is sufficiently high. In general, we observe a trade off between risk sharing and ambiguity. When the aversion to ambiguity is small enough, the benefits of risk sharing dominate and more complete contracts are efficiently chosen. When aversion to ambiguity is large, liquidated damages contracts are chosen. It is natural to ask whether the efficiency of liquidated damages contracts obtains in the standard state-space approach. We show in Proposition 5 that it does not.

Efficiency arguments for default clauses in contracts in the economics literature date back to Shavell (1980), and are elaborated in Che and Chung (1999). The efficiency of such contracts rest on investment and default incentives. These papers treat risk neutral parties in the absence of ambiguity, so there is no contractual rationale for risk sharing or ambiguity aversion.

In contrast, we treat risk averse and ambiguity averse parties. Chung (1991) pointed out the difficulty of simple contracts being efficient when both parties are risk averse. Nevertheless, we find that liquidated damages contracts can be efficient when coupled with ambiguity aversion.

Motivations for liquidated damages given in the legal literature are more in line with our approach. As argued by Hillman (2000, p. 732):

"Because people do not like ambiguity, contracting parties may prefer the safety of a liquidated damages provision over the uncertainty of expectancy damages."

Similarly, Goetz and Scott (1977, p. 557) explain:

"The expected cost of establishing true losses under conventional damage measures will thus induce parties who face uncertain or unprovable anticipated losses to negotiate stipulated damage agreements."

The efficiency of liquidated damages contracts in our model rests on the aversion to ambiguity being sufficiently pronounced to induce the parties to forgo risk sharing opportunities in default states. In general, however, efficient contracts exploit risk sharing opportunities in non-default states.

The paper is organized as follows. In section 2, we set up a formal test-based language in which contracts are specified and derive a representation for preferences in the absence of ambiguity. Next, in Section 3, we develop the concept of contractual ambiguity, and derive preferences over ambiguous contracts using an ε-contamination model. In Section 4 we apply our model to give some results on liquidated damages contracts. In Section 5 we discuss the implications of our analysis and its relationship to the existing literature on incomplete contracts and bounded rationality.

2 Tests, actions and contracts

We consider two parties $i = 1, 2$. Following the approach of Blume et al. (2006), we assume that both players have access to a non-empty set of primitive test propositions $T_0 = \{t_1, ..., t_K\}$ and a set of actions A_0. Let T denote the closure of T_0 under conjunction ($\wedge$) and negation ($\neg$). We use $t \vee t'$ as an abbreviation for $\neg(\neg t \wedge \neg t')$.

For the semantics, we follow Blume et al. (2006) and use a state space that is equivalent to their set of atoms over primitive tests. We set $S^i = S = \{0, 1\}^K$ for $i = 1, 2$, with $|S| = 2^K$.

Hence a state s is a vector of zeroes and ones (a binary number) where the k^{th} component of s denotes the result of test t_k in state s, with 0 (respectively, 1) corresponding to the result of the test is 'not true' (respectively, 'true'). We use $r_k(s)$ to denote the k^{th} component of s.

For convenience we denote by s_0 the state $(0, 0...0)$, by s_1 the state $(0, 0...0, 1)$, and so on up to $s_{[|S|-1]}$ for the state $(1, ..., 1)$.

A test interpretation is a function $\pi : T \to 2^S$, where $\pi(t)$ is the set of states in which the test t is true. The state space $S = \{0,1\}^K$ induces a test interpretation constructed as follows. For each t_k in T_0, set $\pi(t_k) = \{s \in S : r_k(s) = 1\}$. The test interpretation is then inductively extended to tests in T by the rule: for any $t, t' \in T$, $\pi(t \wedge t') = \pi(t) \cap \pi(t')$, and $\pi(\neg t) = S - \pi(t)$.

Conversely, each state $s \in S$ can be identified with a test $t(s) = t_1(s) \wedge ... \wedge t_K(s) \in T$ defined as follows. For each $k = 1, ..., K$ let:

$$t_k(s) = \begin{cases} t_k \text{ if } r_k(s) = 1; \\ \neg t_k \text{ if } r_k(s) = 0. \end{cases}$$

By construction $\pi(t(s)) = \{s\}$ meaning the test $t(s)$ is satisfied only at the state s.

We are interested in the set of *contracts* C, which are constructed inductively from the set of actions A_0 and the set of tests T by taking the closure under the 'if-then-else' construction. That is, we take each a in A_0 to be a contract, and then we require, for any pair of contracts c and c' and any test t in T, that the program '**if** t **then** c **else** c'' should be a contract in C. This contract requires the parties to follow the course of action as determined by contract c if test t is satisfied and follow the course of action as determined by contract c' otherwise.

For any $a \in A_0$, f_a is the unconditional act $f_a(s) = a$ for all $s \in S$. Fix a pair of contracts c and c' in C with associated state-contingent actions f_c and $f_{c'}$. Then for any test t in T, the state-contingent action associated with the contract $c'' =$ '**if** t **then** c **else** c'' is given by $f_{c''}(s) = f_c(s)$ if $s \in \pi(t)$, and $f_{c''}(s) = f_{c'}(s)$ if $s \notin \pi(t)$. It follows from the inductive construction of the set of contracts above that for each c in C, there is an associated 'state-contingent' act $f_c : S \to A_0$.

Conversely, for a given act $f : S \to A_0$, we can define the associated canonical contract c_f with an exhaustive specification given by

$$\textbf{if } t(s_0) \textbf{ then } f(s_0) \textbf{ else if } t(s_1) \textbf{ then } f(s_1) \textbf{ else ...}$$
$$\textbf{else if } t\left(s_{[|S|-2]}\right) \textbf{ then } f\left(s_{[|S|-2]}\right) \textbf{ else } f\left(s_{[|S|-1]}\right)$$

Consider now the individuals' 'ambiguity-free' preferences defined over the set of contracts C. These should be interpreted as the players' preferences over contracts in the absence of any consideration of possible disputes. That is, these are the preferences each player would have, under the assumptions that the other party has the same understanding of the tests used to specify the contract, and that the contract is implemented according to this shared understanding. In the next section, we consider the possibility of a dispute arising from different interpretations of 'ambiguous' tests.

We assume these preferences admit a representation of the following form: there exists for each state s in S a continuous utility function $u_s^i : A_0 \to \mathbb{R}$, such that the following additively-separable function represents the ambiguity-free preferences of individual i:

$$U^i(c) = \sum_{s \in S} u_s^i(f_c(s)). \tag{1}$$

We show in the Appendix that the additive separability across states embodied in expression (1) arises by requiring the preferences to satisfy (along with some other standard properties) the analog of Savage's sure-thing principle. However, as is well-known (see for example Karni, 1985), unless there is some exogenously given structure on the payoffs and their utility, in this formulation, as far as the "ambiguity-free" preferences represented by $U^i(.)$ are concerned, one cannot separate the probability of the state obtaining from the state-dependent utility. One cannot even determine the level of state-dependent utility. More precisely, it is the only the *change* in the state-dependent utility resulting from a change in the action taken in that state that is determined up to a positive scalar. From expression (1) it follows that if $u_s^i(\cdot)$ is a state-dependent utility can be used for the representation in (1) then so can any function $\tilde{u}_s^i(a) = \alpha \tilde{u}_s^i(a) + \beta_s$, with $\alpha > 0$. But notice that for any pair of actions a and a' and any pair of states s and s', we have:

$$\frac{\tilde{u}_s^i(a) - \tilde{u}_s^i(a')}{\tilde{u}_{s'}^i(a) - \tilde{u}_{s'}^i(a')} = \frac{u_s^i(a) - u_s^i(a')}{u_{s'}^i(a) - u_{s'}^i(a')}.$$

We thus define the following equivalence class for state-dependent utilities.

Definition 1 *The state-dependent utility functions* $(u_s)_{s \in S}$ *and* $(\tilde{u}_s)_{s \in S}$ *are* cardinally equivalent *if there exists a positive scalar* $\alpha > 0$ *and vector of constants* $(\beta_s)_{s \in S}$*, s.t.* $\tilde{u}_s(a) \equiv \alpha u_s(a) + \beta_s$ *for all* s *in* S*.*

In what follows, we shall restrict attention to individuals whose preferences in the absence of ambiguity admit a state-dependent expected utility representation of the form given in (1). We shall identify such a preference relation by its state-dependent expected utility representation.

Definition 2 *Let $\mathcal{U}$ denote the set of* state-dependent expected utility functions *defined on the set of contracts C that take the form given in* (1)*.*

3 Introducing Ambiguity

Because we have chosen formally identical state spaces for the players, the test interpretation of each player and the language of each player are identical. The distinction and the source of disputes thus arises from the interaction between syntax and semantics. Disputes arise from the players disagreeing about which tests have been satisfied, or, in a semantic rendition, which state of nature applies. In this section we first introduce ambiguity by way of ambiguous tests and show how this makes some contracts 'ambiguous'. We then develop a model of ambiguity averse decision-makers.

3.1 Conclusive and ambiguous tests and contracts

In this section we introduce the notion of ambiguous tests. This notion will be based on a primitive notion of conclusiveness of a test. The idea of conclusiveness of a test t for an individual i with respect to individual $(3-i)$ is that if she finds herself in a position where she assesses that t is satisfied, then she is sure that individual $(3-i)$ will assess t as satisfied also. The set of conclusive tests for individual i will be denoted by T_C^i. We presume that the individuals are mutually cognizant of T_C^1 and T_C^2. The test t is *unambiguous* if it is conclusive for both individuals. The set of unambiguous tests for individuals 1 and 2 is denoted $T_U = T_C^1 \cap T_C^2$.[2]

To ensure that the sets of conclusive tests match our intuition, we assume that T_C^1 and T_C^2 exhibit the following properties.

Properties of Conclusive Tests: For any pair of tests t and t' in T :

(i) the test $t \vee \neg t$ is in T_C^i (that is, all tautologies are conclusive);

(ii) if the test t is in T_C^i then the test $\neg t$ is in $T_C^{(3-i)}$(that is, the negation $\neg t$ is conclusive for the individual $(3-i)$ with respect to i);

[2] In a model with more than two individuals, it would be necessary to use the notation $T_U^{1,2}$, since the set of unambiguous tests is specific to the given pair $(1, 2)$. In the two-player model presented here, this is unnecessary and superscripts are dropped for simplicity.

(iii) if the tests t and t' are in T_C^i, then the test $t \vee t'$ is in T_C^i (that is, T_C^i is closed under disjunction);

(iv) if $\pi(t) = \pi(t')$ and the test t is in T_C^i, then the test t' is also in T_C^i (two semantically equivalent propositions are either both conclusive or neither)

The next proposition shows that these properties guarantee that any test satisfied in every state or in no state is unambiguous and also that the set of unambiguous tests is closed under negation and conjunction.

Proposition 1 *Fix T_C^1 and T_C^2. If T_C^1 and T_C^2 satisfy the properties of conclusive tests then for each pair of tests t and t' in T:*

(i) *if $\pi(t) = S$ or $\pi(t) = \varnothing$ then $t \in T_U$;*

(ii) *if t,$t' \in T_U$, then* (a) $\neg t \in T_U$ *and* (b) $t \wedge t' \in T_U$

Proof. (i) First, let $\pi(t) = S$. By property *(i)*, the test $t \vee \neg t$ is in T_C^i for $i = 1, 2$. Since $\pi(t \vee \neg t) = S = \pi(t)$, it follows by property *(iv)* and the definition of an unambiguous test that $t \in T_U$. Next, let $\pi(t) = \varnothing$. Then, $\pi(\neg t) = S$, so as just shown above using properties *(i)* and *(iv)*, the test $\neg t$ is in T_U. Then, by property *(ii)*, the test $\neg\neg t$ is in T_C^i for $i = 1, 2$, and so by the definition of an unambiguous test, the test $\neg\neg t \in T_U$. Noting that $\pi(t) = \pi(\neg\neg t)$, it follows from property *(iv)* that $t \in T_U$.

(ii) Let t,$t' \in T_U$. Then, t,$t' \in T_C^i$ for $i = 1, 2$. (a) Consider $\neg t$. By property *(ii)* and the definition of an unambiguous test, $\neg t \in T_U$; (b) Consider $t \wedge t'$. Observe that $\pi(t \wedge t') = \pi(\neg(\neg t \vee \neg t'))$. By properties *(ii)* and *(iii)* and the definition of an unambiguous test, the test $\neg(\neg t \vee \neg t') \in T_U$. Thus applying property *(iv)*, $t \wedge t' \in T_U$. ■

Given that the two individuals are mutually cognizant of T_C^1 and T_C^2 and that they satisfy the four properties listed above, it follows that for any contract of the form '**if** t **then** a **else** a',' if t is an unambiguous test then both individuals anticipate that they will agree whether or not test t has been satisfied. Thus they will agree whether or not the contract calls for action a or for action a'. Suppose, however, the test is conclusive only for individual i and is not conclusive for individual $(3-i)$. Then i anticipates that, when she has assessed test t is satisfied individual $(3-i)$ will that agree the contract calls for action a. However, individual $(3-i)$ believes when he has assessed test t is satisfied, there may be a disagreement with i about whether the contract calls for action a or a'. Conversely, it follows from property (ii) that individual $(3-i)$ anticipates that when he has assessed test t is not satisfied, individual i will also have assessed that test t is not satisfied and so will agree that the contract calls for action a'. Individual i, on the other hand, anticipates that when she has assessed that test t is not satisfied there may be a disagreement with individual $(3-i)$ about whether the contract calls for action a or a'.

We can use the test interpretation to derive the set of unambiguous events.

Definition 3 *The set of* unambiguous events $\mathcal{E}_U \subseteq 2^S$ *is given by:*

$$\mathcal{E}_U = \{E \subseteq S : \pi(t) = S \text{ for some } t \in T_U\} \enspace .$$

The set of ambiguous events $\mathcal{E}^A = 2^S - \mathcal{E}_U$.

Lemma 2 *The set of unambiguous events $\mathcal{E}_U$ is an algebra of subsets of S that contains S and $\varnothing$. That is, it is closed under taking complements and intersection.*

Proof. Assertion *(i)* of Proposition 1 implies that $\mathcal{E}_U$ contains S and $\varnothing$. Consider any pair of unambiguous events E and E' in $\mathcal{E}_U$. Since they are unambiguous events, there must exist tests t and t' in T_U, such that $\pi(t) = E$ and $\pi(t') = E'$. Assertion *(ii)* of Proposition 1 states that T_U is closed under negation and conjunction, so the tests $\neg t$ and $t \wedge t'$ are also in T_U. Since $\pi(\neg t) = S - E$ and $\pi(t \wedge t') = E \cap E'$, the events $S - E$ and $E \cap E'$ are unambiguous. ■

For each $s \in S$, and for each individual i, we can derive from the set of unambiguous tests for individual i, the collection of possible states the other individual ($(3-i)$ may have determined as having obtained as follows.

Definition 4 (*Possibilty of Dispute Set for* i) *Suppose $T_C^i \subset T$, is the set of conclusive tests for individual i. For each s in S, define the* possibility-of-dispute for i *associated with state s to be:*

$$D^i(s) := \{s' \in S : \text{for each } t \in T_C^i,\ s \in \pi(t) \Rightarrow s' \in \pi(t)\}.$$

By construction, the set $D^i(s)$ comprises those states that cannot be distinguished from s by a conclusive test for i being satisfied. Clearly, $s \in D^i(s)$ for each $s \in S$, so $D^i(s) \neq \varnothing$ for each $s \in \Sigma$. We will refer to $\left\{D^i(s)\right\}_{s\in S}$ as the *possibility of disputes for* i.

For each $s \in S$ we can define $E(s)$, the smallest unambiguous event containing s, by

$$E(s) := \bigcap_{E \in \{F \in \mathcal{E}_U : s \in F\}} E.$$

We have the following facts which shows that coarsest common-refinement of $\left\{D^1(s)\right\}_{s\in S} \cup \left\{D^2(s)\right\}_{s\in S}$ is the finest unambiguous partition of S. More specifically, for each state s, the possibility-of-dispute set for i, $D^i(s)$, is a subset of $E(s)$ with equality, if and only if $D^1(s) = D^2(s)$, and $D^i(s)$ is a singleton if and only if the test $t(s)$ associated with the state s is an conclusive test for i.

Lemma 3 *For each $s \in S$: (a) $D^i(s) \subseteq E(s)$ and $D^1(s) = D^2(s) \Rightarrow D^i(s) = E(s)$; (b) $D^i(s) = \{s\}$ if and only if $t(s) \in T^i_C$.*

Proof. (a) First we show $D^i(s) \subseteq E(s)$. Suppose that $s' \in D^i(s)$, but $s' \notin E(s)$. Observe that $E(s) \neq \varnothing$. Hence, there must be some $E \in \{F \in \mathcal{E}_U : s \in F\}$, and $s' \notin E$. Since $E \in \mathcal{E}_U$, there is a test $t \in T_U$ such that $\pi(t) = E$. Also, $s \in E(s)$. Since $s' \in D^i(s)$, it follows from the definition of $D^i(s)$ that $s' \in \pi(t) = E$, which is a contradiction. Hence, we conclude that $D^i(s) \subseteq E(s)$.

Next we show that $E(s) \subseteq D^i(s)$ whenever $D^1(s) = D^2(s)$. Suppose that $s' \in E(s)$, but $s' \notin D^1(s) = D^2(s)$. Then there is some test $t \in T_U$ such that $s \in \pi(t)$ but $s' \notin \pi(t)$. Then $\pi(t)$ is an unambiguous event containing s but not containing s'. Hence $E(s) \subseteq \pi(t)$,and $s' \notin E(s)$, which is again a contradiction. Hence we conclude that $E(s) \subseteq D^i(s)$.

(b) (If) Clearly, $\{s\} \subseteq D^i(s)$ from the definition of $D^i(s)$. Next, since $t(s) \in T^i_C$ and $\pi(t(s)) = \{s\}$, it follows that if $s' \neq s$, then $s' \notin D^i(s)$, that is, $D^i(s) \subseteq \{s\}$.

(Only-if) Since $D^i(s) = \{s\}$, it follows that for each $s' \neq s$, there is a test $t' \in T^i_U$ such that $s \in \pi(t')$ and $s' \notin \pi(t')$. Since T^i_C is closed under conjunction by assertion *(ii)* of Proposition 1, we can take the conjunction of these tests over $S - \{s\}$ to obtain a conclusive test for i, $t^* \in T^i_C$ that excludes everything but s, that is, $\pi(t^*) = \{s\}$. Since $\pi(t(s)) = \{s\} = \pi(t^*)$, it follows from property (iv) of the conclusive test set T^i_C that $t(s) \in T^i_C$. ■

Notice that if a contract is measurable with respect to the *unambiguous partition*, $\left\{E^i(s)\right\}_{s\in S}$ although the individuals might disagree about the actual state that has obtained, they will never disagree about which action the contract prescribes. Hence such contracts are viewed as *unambiguous.*

Definition 5 *A contract is* unambiguous *if for all for all $s,\ s' \in S$, $E(s) = E(s') \Rightarrow f_c(s) = f_c(s')$. We denote by C_U the set of unambiguous contracts.*

3.2 Preferences under ambiguity

We now develop a model the effects of ambiguity has on preferences over contracts. Consider an individual i whose preferences over contracts, in the absence of ambiguity, admit a representation $U^i \in \mathcal{U}$. When individual i believes that the state is s, she considers it possible that the other party may believe any element of $D^i(s)$ has obtained. Hence in terms of a given contract c, this possibility of dispute generates *ambiguity* about the action that will actually be implemented. Depending upon which interpretation is followed, the action might conceivably be any member of the set $\{f_c(s') : s' \in D^i(s)\}$.

We assume that individuals anticipate that a dispute will lead to a 'war of attrition' game in which each player's equilibrium payoff is equal to their security, in this case, the outcome associated with the other player's interpretation.[3] That is, if player i sees state s and $(3-i)$ sees s' then player i's expected payoff in the war-of-attrition equilibrium is $\min\left\{u_s^i(f_c(s)), u_s^i(f_c(s'))\right\}$. If the dispute set $D(s)$ contains only two elements, then the player can evaluate the result of a dispute directly.

More generally, given that disputes are resolved by a war of attrition, individual i can do no worse than accept the least favorable action implied by the contract in the set of possible interpretations of the tests by $(3-i)$ at s, that is, in the set $\{f_c(s') : s' \in D^i(s)\}$.

Hence, one possible way to model the potential loss from a dispute and the resulting war of attrition is to assign a decision weight to this worst-case outcome. This reasoning corresponds to one of the most commonly applied models of ambiguity averse preferences, the ε-contamination

[3] We thank Roger Myerson for the suggestion that disputes could be modelled as wars of attrition.

model.[4]

If we let ε_s^i be the decision-weight she assigns to the ambiguity she faces in state s, then her ε-contaminated subjective expected utility $V_s^i(c)$ of contract c in state s is given by

$$\left(1-\varepsilon_s^i\right) u_s^i\left(f_c\left(s\right)\right)+\varepsilon_s^i \min_{s'\in D^i(s)} u_s^i\left(f_c\left(s'\right)\right). \tag{2}$$

In what follows, we refer to an increase in ε_s^i as a greater aversion of i to ambiguity in state s. We let $V^i(c)=\sum_{s\in S} V_s^i(c)$ denote her *ex ante* expected utility from contract c. A contract c is *ex ante efficient* if there is no other contract c' such that $V^i(c')\geq V^i(c)$ for $i=1,2$, with a strict inequality for some i.

Depending on the degree of concavity of the state-dependent utility functions $u_s^i s$ compared to the decision-weights $\varepsilon_s^i s$, the ambiguity may lead players to prefer incomplete risk sharing to possibly ambiguous contracts. This point may be illustrated with reference to the idea of liquidated damages.

4 Liquidated damages

To be effective, a contract must specify some sanction to be applied if one or other party fails to perform their obligations. In some cases, this is a relatively simple matter: failure to perform may be held to nullify the contract. In other cases, however, failure by one party to perform an obligation may cause damage to the other.

For concreteness, let us consider an example where a supplier contracts with a builder to deliver materials on a given date. However, under certain conditions (expressed as tests), the supplier may be unable to deliver, and may default, declaring *force majeure*. Failure to deliver on time may force the builder to source the supplies elsewhere at high cost, or to delay the project. Thus, neither nullifying the contract nor requiring (delayed) performance is an adequate remedy. The costs of failure will depend on a variety of factors, which may be represented by tests. For example, rainy weather might halt construction with the result that the supplier's default causes no additional cost. In other cases, the default may occur at a crucial point in the project, creating unusually large damages.

[4] The approach here can viewed as a state-dependent extension of Kopylov (2008).

In the absence of bounds on rationality, the parties could agree on a contract that listed all possible default states, and specified a payment to be made in each case. The bargaining solution in this case, derived from the state-dependent preferences of both parties, will be referred to as the first best. However, with ambiguity arising from bounded rationality, the first best may not be attainable.

One solution is for the contract to specify that the defaulting party should compensate the other party to an amount depending on the amount of their loss. In the event of a dispute over the magnitude of the loss, a court or other external arbiter will determine the payment.

Another possibility is that of liquidated damages, in which the payment for a specific breach is fixed in the contract, without reference to the actual losses suffered by the injured party. We will look at the liquidated damages setting and give some results on when liquidated damages contracts are efficient.

We begin by assuming that the test set T_0 includes the test t^d, interpreted as 'the state is such that party 1 must default' and a set of tests $\hat{T} \subset T$, that are relevant to the contract in the absence of default. We assume that t^d is unambiguous and that the members of the set of tests $\hat{T} \subset T$, are also unambiguous. That is, the only potential disputes relate to the consequences of default, and not to the question of whether party 1 has in fact defaulted. Without loss of generality we take $t^d = t_K$, hence the default event $\pi\left(t^d\right) = \{s \in S : r_K(s) = 1\}$.

The action set $A = \hat{A} \times Y$ is the Cartesian product a set of actions $\hat{A}$ relevant to the performance of the contract and a set of payment actions (monetary transfers) $Y = [-M, M]$. The set $\hat{A}$ is assumed to include a default action a_0. Actions $y \in Y$ are interpreted as 'party 1 pays y dollars to party 2. Actions $\hat{a} \in \hat{A} - \{a_o\}$ are unavailable or prohibitively costly to party 1 in the event of default. Thus, any feasible contract c must satisfy

$$\begin{array}{ll} f_c(s) = (a_c(s), y_c(s)) \in \hat{A} \times Y & \text{if } r_K(s) = 0 \\ f_c(s) = (a_0, y_c(s)) \in \{a_0\} \times Y & \text{if } r_K(s) = 1 \end{array}.$$

That is, the contract specifies a set of actions to be performed, and payments to be made, in the absence of default and a set of payments to be made in the presence of default. The payment

$y_c(s)$ for any $s \in \pi(t_K)$ (that is, a payment made in the presence of default) is referred to as a damages payment.

We further assume that, for each party $i = 1, 2$ in each state default state $s \in \pi(t_K)$, the preferences over $(\hat{a}, y)$ are quasi-linear with respect to the damages payment and so the utility in each state may be expressed in the form

$$u_s^i(\hat{a}, y) = v_s^i\left(w_s^i(\hat{a}) + (-1)^{(2-i)}y\right),$$

where v_s^i is a state-dependent utility function over wealth, assumed strictly increasing ($\left(v_s^1\right)' > 0$) and strictly concave ($\left(v_s^1\right)'' < 0$), and $w_s^i(\hat{a})$ is the monetary equivalent value to party i of the action $\hat{a}$ performed in state s.

Notice that for any default state $s \in \pi(t_K)$, $(s_{-K}, 0) = (s_1, s_2, ..., s_{K-1}, 0)$ is the state that would have obtained if party 1 had not defaulted with all the results of all other basic test in T_0 unchanged. In this counter-factual state, the contract would have called for action $a_c(s_{-K}, 0)$. Hence, $w_s^i(a_c(s_{-K}, 0)) - w_s^i(a_0)$ may be interpreted as the loss incurred by party i in state s, as a consequence of the default.

In the absence of ambiguity, a Pareto optimal contract c^* must satisfy the Borsch condition for efficient risk-sharing, that is, the marginal rate of substitution between any pair of state-contingent payoffs must be the same for both individuals. Formally, for any pair of default states $s, \tilde{s} \in \pi(t_K)$.

$$\frac{\left(v_s^1\right)'\left(w_s^1(a_0) - y_{c^*}(s)\right)}{\left(v_{\tilde{s}}^1\right)'\left(w_{\tilde{s}}^1(a_0) - y_{c^*}(\tilde{s})\right)} = \frac{\left(v_s^2\right)'\left(w_s^2(a_0) + y_{c^*}(s)\right)}{\left(v_{\tilde{s}}^2\right)'\left(w_{\tilde{s}}^2(a_0) + y_{c^*}(\tilde{s})\right)}$$

A contract satisfying this condition will be referred to as a first-best contract. Since the contract is unambiguous in the absence of default, the first-best contract will, in general, be unambiguous if and only if the set of unambiguous tests is rich enough to distinguish any pair of states $s, \tilde{s} \in \pi(t_K)$ such that either $w_s^1(a_0) \neq w_{\tilde{s}}^1(a_0)$ or $w_s^2(a_0) \neq w_{\tilde{s}}^2(a_0)$.

Suppose, however, that tests relevant to the effects of default on the welfare of party 1 (the defaulting party) are ambiguous.

In this case, we may consider the case of a contract with damages dependent on losses to party

2. Since the cardinality of $\pi(t_K)$ is finite it follows that the set

$$L^2 = \left\{\ell \in [0, M] : \exists s \in \pi(t_K) \text{ s.t. } \ell = w_s^2(a_c(s_{-K}, 0)) - w_s^2(a_0)\right\}$$

is also finite. Moreover, for each $\ell \in L^2$, there exists a test $t_\ell \in T$ that is satisfied if and only if default occurs, and the associated loss for party 2 is ℓ, that is, on the event $\pi(t_\ell) = \pi(t_K) \cap \left\{s : w_s^2(a_c(s_{-K}, 0)) - w_s^2(a_0) = \ell\right\}$. The members of the set of events $\left\{\pi(t_\ell) : \ell \in L^2\right\} \cup \{\pi(\neg t_K)\}$, are mutually exclusive and exhaustive, and therefore constitute a partition of the state space. Any contract c can be amended in a way to make it is a loss-dependent damages contract $\hat{c}$, by restricting it to be measurable with respect to this partition. That is, $\hat{c}$ may then be specified as

$$f_{\hat{c}}(s) = \begin{cases} f_c(s) & s \in \pi(\neg t_K) \\ y_{\hat{c}}(\ell) & s \in \pi(t_\ell) \end{cases}$$

where $y_{\hat{c}} : \mathbb{R}_+ \to [0, M]$ is a function relating the loss borne by party 2 to the associated damages payment from party 1. Note that we do not require $y_{\hat{c}}(\ell) = \ell$. That is, the damages payment from party 1 to party 2 need not be equal to the loss incurred by party 2. Depending on the risk-sharing properties of the contract and on the state-dependent preferences of party 1, the damages payment to party 2, $y_{\hat{c}}(\ell)$, may be less than, equal to or greater than the loss ℓ incurred by party 2.[5]

The tests t_ℓ may still be ambiguous. For example, the parties may disagree over what items should be counted as losses arising from default and how they should be valued. Thus, such contracts are likely to, and regularly do, produce disputes.

If losses are ambiguous, and dispute costs are high, parties may prefer a *liquidated damages contract*, with a specified payment $\bar{y}$. The required test set is then the minimal set $\{t_K\}$ and the contract simply requires

[5] In general, risk-sharing would imply that the damages payment should be less than the loss. In the model presented here, losses are the result of *force majeure* rather than discretionary options. Hence, there is no incentive-based reason for exemplary or punitive damages. However, consideration of the state-contingent preferences of party 1 suggests instances where risk-sharing may imply a payment larger than the loss. Suppose that high-losses to party 2 occur when the good is in high demand and subject to constrained supply. Then party 1, having defaulted as a result of inability to supply on time may be able to sell the good at a high price and therefore (involuntarily) benefit from default.

$$\textbf{if } \neg t_K \textbf{ then } c \textbf{ else } (a_0, \bar{y})$$

Hence,

$$f_{\bar{c}}(s) = \begin{cases} f_c(s) & s \in \pi(\neg t_K) \\ (a_0, \bar{y}) & s \in \pi(t_K) \end{cases}$$

That is, either the contract applicable in the absence of default is implemented or the default action a_0 is undertaken and party 1 pays to party 2 the liquidated damage sum $\bar{y}$. As long as the test t_K is unambiguous, so is the liquidated damages contract.

In general, there will be gains from risk sharing across states. When the aversion to ambiguity is small across states, efficient contracts will involve risk sharing even at the ambiguous default states. However, when the aversion to ambiguity is sufficiently large, there will be no risk sharing across default states, i.e., all efficient contracts will be liquidated damages contracts. We formally give this result in the next proposition. For simplicity, we presume that the possibility of dispute set $D^i(s)$ is the same for each default state $s \in \pi(t_K)$.

Proposition 4 *Suppose that for $i = 1, 2$ and all $s \in \pi(t_K)$, $D^i(s) = \pi(t_K)$. There is an $\varepsilon < 1$ such that: if $\varepsilon_s^i > \varepsilon$ for all $i \in \{1, 2\}$ and $s \in \pi(t_K)$, then every ex ante efficient contract is a liquidated damages contract.*

Proof. Suppose a contract c is not a liquidated damages contract. Then, let $\overline{y}$ denote the maximal payment over default states under this contract, i.e., $\overline{y} = \max_{s \in \pi(t_K)} y_c(s)$, and let $\underline{y}$ denote the minimal payment over default states, i.e., $\underline{y} = \min_{s \in \pi(t_K)} y_c(s)$. Then, $\overline{y} > \underline{y}$. We will show that provided ε is large enough, we can increase the welfare of both parties 1 and 2 by marginally increasing $\underline{y}$ and marginally decreasing $\overline{y}$.

For, this, we define:

$$\begin{aligned} a &= \max_{i \in \{1,2\}} \max_{s \in \pi(t_K)} (v_s^i)'(w_s^i(a_0) - M); \\ b &= \max_{i \in \{1,2\}} \max_{s \in \pi(t_k)} (v_s^i)'(w_s^i(a_0) + M); \\ \varepsilon &= \frac{x}{x+y}. \end{aligned}$$

We presume in what follows that: $\varepsilon_s^i > \varepsilon$ for all $i \in \{1,2\}$ and $s \in \pi(t_K)$. Let $\underline{S} = \{s \in \pi(t_K) : \underline{y} = y_c(s)\}$, $\overline{S} = \{s \in \pi(t_K) : \overline{y} = y_c(s)\}$, and $S^* = \pi(t_K) - (\overline{S} \cup \underline{S})$. Then the ex ante expect utility of 1 can be written as:

$$\begin{aligned} V^1(c) &= \sum_{s \in S - \pi(t_K)} V_s^1(c) + \sum_{s \in \underline{S}} \left[(1-\varepsilon_s^1) v_s^1(w_s^1(a_0) - \underline{y})\right] + \sum_{s \in S^*} \left[(1-\varepsilon_s^1) v_s^1(w_s^1(a_0) - y_c(s))\right] \\ &\quad + \sum_{s \in \overline{S}} \left[(1-\varepsilon_s^1) v_s^1(w_s^1(a_0) - \overline{y})\right] + \sum_{s \in \pi(t_K)} \left[\varepsilon_s^1 v_s^1(w_s^1(a_0) - \overline{y})\right]. \end{aligned}$$

We consider a marginal change to $\underline{y}$ and $\overline{y}$ such that $d\underline{y} = -d\overline{y}$. For such a change:

$$\begin{aligned} dV^1(c) &= -\sum_{s \in \underline{S}} \left[(1-\varepsilon_s^1)(v_s^1)'(w_s^1(a_0) - \underline{y}) d\underline{y}\right] + \sum_{s \in \overline{S}} \left[(1-\varepsilon_s^1)(v_s^1)'(w_s^1(a_0) - \overline{y}) d\underline{y}\right] \\ &\quad + \sum_{s \in \pi(t_K)} \left[\varepsilon_s^1 (v_s^1)'(w_s^1(a_0) - \overline{y}) d\underline{y}\right]. \end{aligned}$$

We let $|A|$ denote the cardinality of a set A. By our choices of a, b and ε, and the facts that $(v_s^1)' > 0$, and $(v_s^1)'' < 0$, we find that:

$$dV^1(c) > \left[-\left|\underline{S}\right|(1-\varepsilon)x + \left|\pi(t_k)\right| \varepsilon y\right] d\underline{y} > \left|\pi(t_k)\right| \left[-(1-\varepsilon)x + \varepsilon y\right] d\underline{y} = 0.$$

By similar reasoning for 2, we obtain $dV^2(c) > 0$. Hence, c cannot be *ex ante* efficient. ■

Here we see that in the case of sufficient aversion to ambiguity over default states, the optimal contract is the liquidated damages contract. The intuition in the maximally pessimistic case is as follows. Since each expects the worst in default states, we can raise the utility of 2 at all default states by raising $\underline{y}$, and simultaneously raise the utility of 1 at all default states by lowering $\overline{y}$. Since this change does not affect utility in any other state, it generates a Pareto improvement. Proposition 4 shows that intuition carries through provided the parties are sufficiently ambiguity averse.

The optimality of the liquidated damages contract is consistent with situations where the parties expect default to result in *ex post* litigation costs that burn up all surplus. By signing a liquidated damages contract, each party commits *ex ante* to abstain from such behavior.

A natural question is whether or not the efficiency of liquidated damages contracts can be maintained in a state space approach. The answer is that it cannot. To see this, we presume that each player will have a partition over $S \times S$ and a probability distribution over those states.

Let $\rho_i(s,t)$ denote the prior probability in $i's$ mind that 1 sees s and 2 sees t. We focus on the event of default which is $E_d = \{(s,t) : s,t \in \pi(t_K)\}$. Then, the probability of a default event in the eyes of i is $\sum_{(s,t)\in E_d} \rho_i(s,t)$. In keeping with our previous analysis, each person's utility and wealth depends only on the state they see so when the state is (s,t) we will write v_s^1, w_s^1, v_t^2, w_t^2. However, we allow the transfer amount y to depend on the state (s,t). The presumption here is that some determination on (s,t) will be made *ex post* and then a transfer occurs. Each player considers each contingency (s,t) as possible when he writes the contract. Under this scenario, the *ex ante* expected utility of 1 and 2 restricted to default states are respectively:

$$\sum_{(s,t)\in E_d} \rho_1(s,t) v_s^1(w_s^1(a_0) - y(s,t))$$

and:

$$\sum_{(s,t)\in E_d} \rho_2(s,t) v_t^1(w_t^2(a_0) + y(s,t))$$

Though typically liquidated damages will not be efficient in this full state-space approach, for a clean result we focus on a case of a common utility function and common prior with a technical assumption about richness of the state-space:

1. (*Common prior*) $\rho_1(s,t) = \rho_2(s,t)$ for all $(s,t) \in E_d$;

2. (*Common state independent utility function*) $v_s^1 = v_s^2 = v$ for all $s \in S$.

3. (*Richness of State-space*) There are s, s' in $\pi(t_K)$ such that either $w_s^1(a_0) \neq w_t^1(a_0)$ or $w_s^2(a_0) \neq w_t^2(a_0)$.

Proposition 5 *Suppose that 1-3 hold. If c is ex ante efficient, then c is not a liquidated damages contract.*

Proof. Suppose c is a liquidated damages contract. By assumption 3, there are two default states s and s' where some player gets a different utility prior to any transfer. We presume, without loss of generality that $w_s^1(a_0) \neq w_{s'}^1(a_0)$. Consider the states (s,s) and (s',s) which are both in E_d. By assumptions 1 and 2, a necessary condition for efficiency is:

$$\frac{v'\left(w_s^1\left(a_0\right) - y_c\left(s,s\right)\right)}{v'\left(w_{s'}^1\left(a_0\right) - y_c(s',s)\right)} = \frac{v'\left(w_s^2\left(a_0\right) + y_c\left(s,s\right)\right)}{v'\left(w_s^2\left(a_0\right) + y_c(s',s)\right)}$$

Since the contract is a liquidated damages contract, $y_c(s, s) = y_c(s', s)$, and so the right hand side of the equality must be 1. However, since $w_s^1(a_0) \neq w_{s'}^1(a_0)$ and $v'' < 0$ (strict concavity), the left hand side cannot equal 1 when $y_c(s, s) = y_c(s', s)$. Since c does not satisfy the necessary condition for efficiency, c is not efficient. ■

5 Concluding comments

We have provided a formal model for incorporating linguistic ambiguity into decision making. The ambiguity in our model arises from the bounded rationality of the players which is expressed as limited abilities to perform tests over the possible contingencies. As a result, players have limited descriptions of the possible states of the world available to them. Even when they use the same language, their interpretations may differ.

Contracting is modelled using a multi-player version of the test-based contingent plans described in Blume et al. (2006). In this context, ambiguity can affect incentives for risk sharing, and the desirability of contracts. In particular, ambiguity may in some cases be handled effectively and efficiently by liquidated damages contracts.

The representation of ambiguity proposed here suggests new approaches to a range of issues in contract theory, and potentially broader applications in agency theory. The standard principal-agent problem is one where contracting is limited to some observable unambiguous characteristics like output, rather than a full set of characteristics including effort levels which may be ambiguous. The framework developed here suggests the possibility of an endogenous choice between contracts over different characteristics, where the choice of the contractual variables chosen depends on the level of ambiguity and potential gains from risk sharing. While this application would require overcoming some new technical details involving the appropriate treatment of tests, the benefit would be the development of a theory of contracting in which the terms of the contract, over which the parties actually bargain, plays the central role.

References

Blume, L.E., D.A. Easley and J.Y. Halpern, (2006), "Redoing the Foundations of Decision Theory," *Tenth International Conference on Principles of Knowledge Representation and*

Reasoning (KR 2006), 14-24.

Board, O. and K-S. Chung, (2007), "Object-Based Unawareness: Axioms." mimeo University of Minnesota.

Board, O. and K-S. Chung, (2009), "Object-Based Unawareness: Theory and Applications." mimeo University of Minnesota.

Che, Y-K., and T-Y. Chung, (1999), Contract Damages and Cooperative Investments, *Rand Journal of Economics*, 30, 84-105.

Chung, T-Y., (1991), Incomplete Contracts, Specific Investments and Risk Sharing, *Review of Economic Studies*, 58, 1031-1042.

Debreu, G., (1960), "Topological methods in cardinal utility theory." In K. J. Arrow, S. Karlin & P. Suppes (Eds.), Mathematical methods in the social sciences, 1959: Proceedings (pp. 16-26). Stanford, CA: Stanford Univ. Press.

Filiz-Ozbay, E. (2010), "Incorporating Unawareness into Contract Theory", mimeo, University of Maryland, College Park.

Goetz, C. J., and Robert E. Scott, (1977), Liquidated Damages, Penalties and the Just Compensation Priniciple: Some Notes on the Enforcement Model and and Theory of Efficient Breach, 77, 554-594.

Grant, S., J. Kline, and J. Quiggin (2011), "Differential awareness and incomplete contracts: a model of contractual disputes" Risk and Sustainable Management Group, Risk and Uncertainty Program Working Paper, University of Queensland.

Hillman, R. A., (2000), The Limits of Behavioral Decision Theory in Legal Analysis: The Case of Liquidated Damages, *Cornell Law Review*, 85, 717-738.

Karni, E., (1985), *Decision Making Under Uncertainty: The Case of State Dependent Preferences.* Harvard University Press, Cambridge, MA.

Kopylov, I., (2008), "Subjective Probability and Confidence," mimeo University of California, Irvine.

Mukerji, Sujoy, (1998), "Ambiguity Aversion and Incompleteness of Contractual Form," *American Economic Review* 88(5), 1207-1231.

Mukerji, Sujoy and Jean-Marc Tallon, (2001), "Ambiguity aversion and incompleteness of financial markets," *Review of Economic Studies*, 68(4), 883-904.

Shavell, S., (1980), Damage Measures for Breach of Contract, *Bell Journal of Economics*, 11, 466-490.

A Axiomatization of state-dependent additively-separable ambiguity-free preferences.

Let $\succsim^i$ denote individual i's (dispute-free) preferences over contracts. Consider the following three axioms.

Ordering Axiom The relation $\succsim^i$ is complete and transitive.

Act-equivalence Axiom For any pair of contracts, c and c' in C, if $f_c = f_{c'}$ then $c \sim c'$.

***Sure-thing Principle*:** For any four contracts c, c', c'' and c''' in C, and any test t in T,

$$\begin{aligned} & \textbf{if } t \textbf{ then } c \textbf{ else } c'' \succsim^i \textbf{if } t \textbf{ then } c' \textbf{ else } c'' \\ \Rightarrow \quad & \textbf{if } t \textbf{ then } c \textbf{ else } c''' \succsim^i \textbf{if } t \textbf{ then } c' \textbf{ else } c'''. \end{aligned}$$

The first axiom is the standard ordering axiom. The second requires any two contracts that induce the same act over actions must come from the same indifference class. This seems natural in a setting in which we assume the agent understands the language in which contracts are written and the logical implications of its terms and attendant requirements. The third axiom is the analog of Savage's *sure-thing principle.*

The fourth axiom is a continuity assumption to ensure a numerical representation of preferences exists. Before stating it, we need to define what it means for a sequence of contracts to converge to a limit. We do this inductively. First, we define the notion of convergence for constant acts directly from the notion of convergence of actions in the set A_0, and then we extend it inductively to all contracts via the 'if..then..else' construction.

Definition 6 (Convergence of Sequences of Contracts) *The (countably infinite) sequence of constant acts $\langle a_n \rangle$ converges to the constant act $\bar{a}$, if the corresponding sequence of actions converge to the corresponding action, that is,* $\lim_{n\to\infty} a_n = \bar{a}$. *For any sequence of tests $\langle t_n \rangle$ and any pair of sequences of contracts $\langle c_n \rangle$ and $\langle c'_n \rangle$, the sequence of contracts $\langle c''_n \rangle$, where $c''_n =$ '**if** t_n **then** c_n **else** c'_n' is said to converge to $\bar{c}'' =$ **if** $\bar{t}$ **then** $\bar{c}$ **else** $\bar{c}'$, if $\langle c_n \rangle$ and $\langle c'_n \rangle$ converge to $\bar{c}$ and $\bar{c}'$, respectively, and there exists finite m, such that $t_n = \bar{t}$ for all $n > m$.*

Continuity of preference can now be expressed in the standard manner of requiring that there are no 'jumps in preference at infinity'.

***Continuity*:** For any pair of sequences of contracts $\langle c_n \rangle$ and $\langle c'_n \rangle$, that converge to $\bar{c}$ and $\bar{c}'$, respectively, if $c_n \succsim^i c'_n$ for all n, then $\bar{c} \succsim^i \bar{c}'$

Finally, we require a minimum amount of non-degeneracy for the preferences with respect to the states in S. Formally, we require at least three states to be 'essential'.

Definition 7 *Fix $\succsim^i$. A state s in S is essential for $\succsim^i$ if there exists a pair of actions a and a' in A_0 and a contract c in C, such that*

$$[\textbf{\textit{if}}\ t(s)\ \ \textbf{\textit{then}}\ a\ \ \textbf{\textit{else}}\ c] \succ^i [\textbf{\textit{if}}\ t(s)\ \ \textbf{\textit{then}}\ a'\ \ \textbf{\textit{else}}\ c].$$

We now have all the pieces for the representation result.

Theorem 6 (State-Dependent Expected Utility Representation) *Fix $\succsim^i$. If there are at least three essential states then the following are equivalent.*

1. *The relation $\succsim^i$ satisfies ordering, act-equivalence, sure-thing principle and continuity.*
2. *There exists for each state s in S a continuous utility function $u_s^i : A_0 \to \mathbb{R}$, such that the following additively-separable function represents $\succsim^i$:*

$$U^i(c) = \sum_{s \in S} u_s^i(f_c(s)) \tag{3}$$

Moreover, the functions $u_s^i(\cdot)$ are unique up to multiplication by a common positive scalar $\alpha > 0$, and the addition of a state-dependent constant β_s.

Proof. Sufficiency of axioms. Consider the preference relation $\succsim_{\mathrm{F}}^i \subset A_0^{|S|} \times A_0^{|S|}$ over acts, induced by $\succsim^i$: $c \succsim^i c'$ implies $f_c \succsim_{\mathrm{F}} f_{c'}$. Consider a pair of acts, $f \succsim_{\mathrm{F}}^i f'$. By construction, there exists a pair of contracts c and c' such that $f_c = f$, $f_{c'} = f'$ and $c \succsim^i c'$. Now for any pair of contracts $\hat{c}$ and $\hat{c}'$, such that $f_{\hat{c}} = f$ and $f_{\hat{c}'} = f'$, it follows from act-equivalence that $\hat{c} \sim c$ and $\hat{c}' \sim c'$, and so by ordering we have $\hat{c} \succsim^i \hat{c}'$. Hence it is enough to obtain a representation $\hat{U}^i(f)$ of $\succsim_{\mathrm{F}}^i$, since we can then set $U^i(c) := \hat{U}^i(f_c)$.

It is straightforward to show that continuity of $\succsim^i$ implies that $\succsim_{\mathrm{F}}^i$ is continuous in the product topology of $A_0^{|S|}$; and that the sure-thing principle for $\succsim^i$ implies that $\succsim_{\mathrm{F}}^i$ satisfies the sure-thing principle for acts: that is, for any four acts f, f' f'' and f''', and any event $E \subset S$, if $f(s) = f''(s)$ and $f'(s) = f'''(s)$ for all $s \in E$, and $f(s) = f'(s)$ and $f''(s) = f'''(s)$ for all $s \notin E$ then $f \succsim_{\mathrm{F}}^i f'$ implies $f'' \succsim_{\mathrm{F}}^i f'''$. Hence by Debreu (1960, Theorem 3) it follows there exists an additive representation for $\succsim_{\mathrm{F}}^i$ as given in (3). Proof of necessity of axioms is omitted. ■